9.1 The Phase Plane: Linear Systems

Qualitative Analysis: concern is with general behavior rather than exact solution or "hard" numbers.

Recall that a critical point is a point where $\vec{x}\,' = 0$. This represents a stationary point since $\vec{x}$ is not changing. Assuming A is nonsingular then $\vec{x}\,' = A\vec{x}$ has only one critical point which is at $\vec{x} = \vec{0}$.

Consider the system

$$\vec{x}\,' = A\vec{x}$$

where A is a 2×2 matrix with constant coefficients.

In this case

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

If we plot x_2 vs. x_1 (t is the parameter) we are creating a <u>phase plane diagram</u>. The curve traced by $(x_1(t), x_2(t))$ as t is varied is called a <u>trajectory</u>.

Assume that A is non-singular [$\det A \neq 0$]. Then $\det(A - rI) = 0$ will have one of the following arrangements of roots

1. r_1, r_2 real & distinct
   - A. $r_1 < r_2 < 0$ or $0 < r_2 < r_1$
   - B. $r_1 > 0$, $r_2 < 0$
2. $r_1 = r_2$ real & repeated
   - A. 2 lin. indep eigenvectors
   - B. 1 lin indep. eigenvector (1 generalized eigenvector)

3. $r_1, r_2 = \lambda \pm i\mu$

   A. $\lambda < 0$ or $\lambda > 0$

   B. $\lambda = 0$ (pure imaginary roots)

CASE 1: $\vec{x} = c_1 \vec{\xi}^{(1)} e^{r_1 t} + c_2 \vec{\xi}^{(2)} e^{r_2 t}$

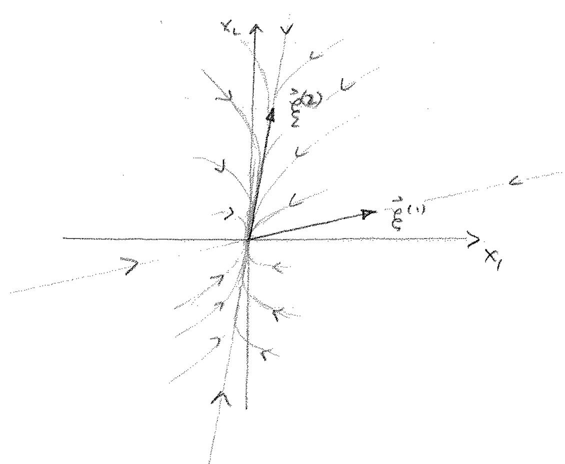
A. $r_1 < r_2 < 0$: In this case $\vec{x} \to \vec{0}$ as $t \to \infty$.

Notice $\vec{x} = e^{r_2 t} [c_1 \vec{\xi}^{(1)} e^{(r_1 - r_2)t} + c_2 \vec{\xi}^{(2)}]$. Since $r_1 < r_2$ we have $r_1 - r_2 < 0$ so as $t \to \infty$ we find

$$\vec{x} \to \underbrace{e^{r_2 t} c_2}_{\text{scalar}} \underbrace{\vec{\xi}^{(2)}}_{\text{vector}} \qquad [\text{assuming } c_2 \neq 0]$$

So the vector solution $\vec{x}$ appears more and more like a multiple of the eigenvector $\vec{\xi}^{(2)}$ associated with r_2.

Thus, all solutions $\vec{x}$ approach $\vec{0}$ as $t \to \infty$ and they approach $\vec{0}$ parallel to $\vec{\xi}^{(2)}$.



In this case the origin is called a <u>node.</u>

$0 < r_2 < r_1$: $x_1, x_2 \to \infty$ as $t \to \infty$.

$\vec{x} \to \vec{0}$ as $t \to -\infty$

$$\vec{x} = e^{r_2 t}\left[c_1 \vec{\xi}^{(1)} e^{(r_1 - r_2)t} + c_2 \vec{\xi}^{(2)}\right]$$

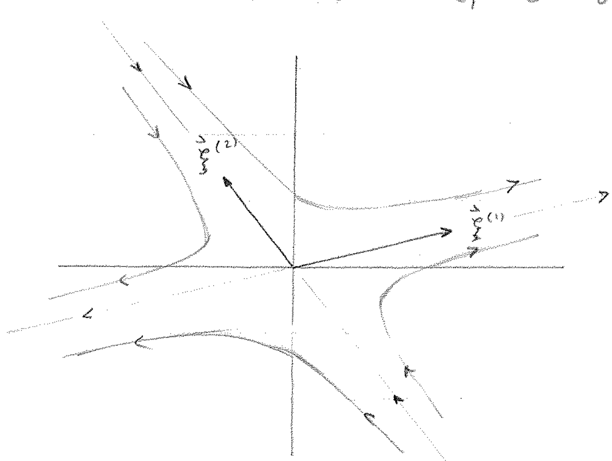
and as $t \to -\infty$ we again find the dominant term is $\underbrace{e^{r_2 t}\, c_2}_{\text{scalar}}\, \underbrace{\vec{\xi}^{(2)}}_{\text{vector}}$

So again trajectories approach $\vec{\xi}^{(2)}$ as they approach the origin. (again, the origin is a node).

B. $r_1 > 0$; $r_2 < 0$

$$\vec{x} = c_1 \vec{\xi}^{(1)} \underbrace{e^{r_1 t}}_{\to \infty} + c_2 \vec{\xi}^{(2)} \underbrace{e^{r_2 t}}_{\to 0}$$

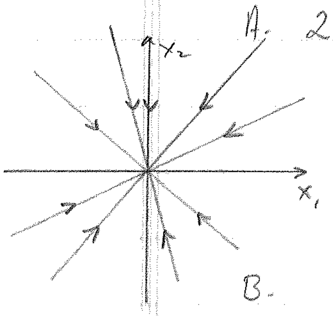
So $\vec{x} \to \vec{0}$ if $c_1 = 0$ otherwise $x_1, x_2 \to \infty$



Get to origin only if initial point is $\propto \vec{\xi}^{(2)}$ (i.e. $c_1 = 0$)

origin is a saddle pt.

CASE 2 : $r_1 = r_2 = r$ real



A. 2 linearly independent eigenvectors: $\vec{x} = \underbrace{(c_1 \vec{\xi}^{(1)} + c_2 \vec{\xi}^{(2)})}_{\text{vector}} \underbrace{e^{rt}}_{\text{scalar}}$

Similar to Case 1 part A except trajectories are straight lines.
(proper node)

B. 1 lin. indep. eigenvector: $\vec{x} = c_1 \vec{\xi} e^{rt} + c_2 [t \vec{\xi} e^{rt} + \vec{\eta} e^{rt}]$

or

$$\vec{x} = [(c_1 \vec{\xi} + c_2 \vec{\eta}) + c_2 \vec{\xi} t] e^{rt}$$

Here, $\vec{\eta}$ is a generalized eigenvector. Examine the bracketed expression:

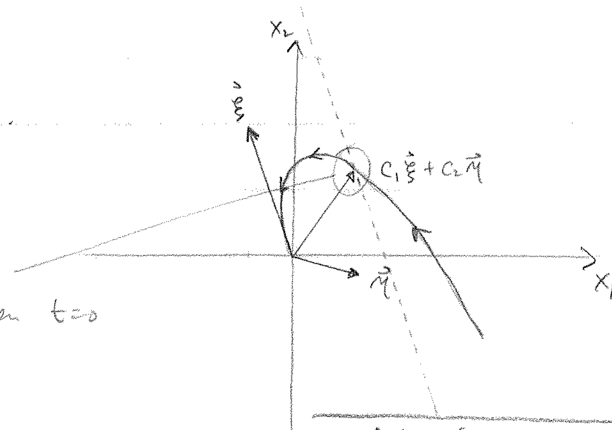
$$(c_1 \vec{\xi} + c_2 \vec{\eta}) + c_2 \vec{\xi} t$$

which is the vector part (and hence the direction part) of $\vec{x}$. e^{rt} is merely a scale factor. When $t=0$ this expression is just

$$c_1 \vec{\xi} + c_2 \vec{\eta}$$

and as $t \to \infty$ it approaches $c_2 \vec{\xi} t$. In fact, it is the parametric description of a line through $c_1 \vec{\xi} + c_2 \vec{\eta}$ and $\parallel$ to $\vec{\xi}$.

Suppose $c_2 > 0$ and $r < 0$.
(See text) ( )



direction here is given by $\vec{x}'$ when $t=0$

$$\vec{x}'|_{t=0} = (c_1 r + c_2) \vec{\xi} + c_2 r \vec{\eta}$$

origin is improper node

CASE 3: $r_1, r_2 = \lambda \pm i\mu$

We will examine the system $\vec{x}' = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix} \vec{x}$

Which will have complex eiginvalues. Since this matrix is symmetric with λ on the diagonal we will find symmetries in our solution not present in the general case. However this much like examining $y = x^2$ to understand some behaviors of parabolas in general.

A.E. $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow$ $\begin{array}{l} x_1' = \lambda x_1 + \mu x_2 \\ x_2' = -\mu x_1 + \lambda x_2 \end{array}$

Since in polar coordinates $r^2 = x_1^2 + x_2^2$ $\quad x_1 = r\cos\theta$

$\tan\theta = x_2/x_1$ $\quad x_2 = r\sin\theta$

we have, upon differentiating the eqs for r^2 and $\tan\theta$ wrt t:

$$2rr' = 2x_1 x_1' + 2x_2 x_2' \rightarrow rr' = x_1 x_1' + x_2 x_2'$$

$$\sec^2\theta\, \theta' = \frac{x_1 x_2' - x_2 x_1'}{x_1^2}$$

so

$$rr' = x_1(\lambda x_1 + \mu x_2) + x_2(-\mu x_1 + \lambda x_2) = \lambda x_1^2 + \lambda x_2^2 = \lambda r^2$$

$$r' = \lambda r$$

$$r = r_0 e^{\lambda t}$$ r_0 is r value at $t=0$

So r will increase with t if $\lambda > 0$ and will decrease if $\lambda < 0$.

$$\sec^2\theta\, \theta' = \frac{x_1(-\mu x_1 + \lambda x_2) - x_2(\lambda x_1 + \mu x_2)}{x_1^2} = -\mu \frac{x_1^2 + x_2^2}{x_1^2}$$

$$\theta' = -\frac{\mu}{x_1^2} r^2\cos^2\theta = -\frac{\mu}{x_1^2} x_1^2 = -\mu$$

So $\quad \theta = -\mu t + \theta_0 \quad$; θ_0 is θ value at $t=0$

$$\begin{cases} r = r_0 e^{\lambda t} \\ \theta = -\mu t + \theta_0 \end{cases}$$

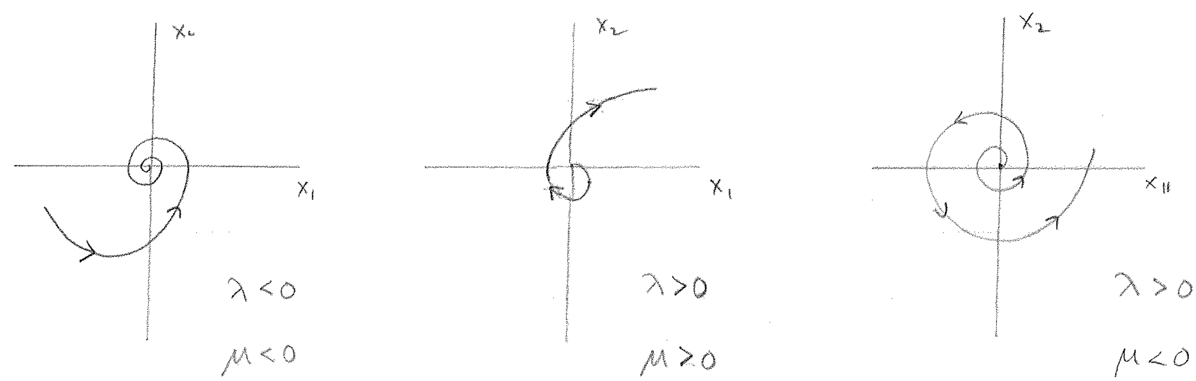
Spiral : $\lambda < 0$: spiral in

$\lambda > 0$: spiral out

$\mu < 0$: counterclockwise

$\mu > 0$: clockwise

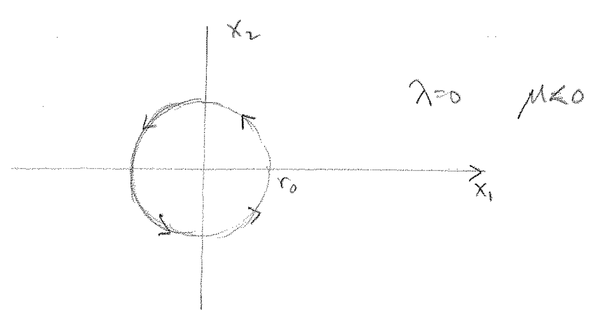
Notice that while θ changes linearly with t, r changes exponentially.



origin is a spiral point

B. Now, if $\lambda = 0$ then $r = r_0$, $\theta = -\mu t + \theta_0$

So we have a periodic motion



Stability

So far we've seen three cases:

1. $\vec{x} \to \vec{0}$ as $t \to \infty$
2. $\vec{x}$ unbounded as $t \to \infty$
3. $\vec{x}$ periodic, i.e. $\exists$ t_1 and t_2 s.t. $t_1 < t_2$ and $\vec{x}(t_1) = \vec{x}(t_2)$

(1) is <u>asymptotically stable</u> since for any $\vec{x}_0$ the solution goes to $\vec{0}$ as $t \to \infty$

(2) is <u>unstable</u>. No matter where one starts, the solution increases without bound as $t \to \infty$

(3) <u>Stable</u> - does not increase without bound.