9.2 Autonomous Systems and Stability

The system $\vec{x}' = A\vec{x}$ is <u>autonomous</u> if none of the entries in A depend explicitly on t. If one or more of the entries of A do depend on t then the system is <u>nonautonomous</u>.

Recall that all of our phase plane portraits contained trajectories that did not cross – i.e. for each point in the plane, at most one trajectory passed through it. This is typical of autonomous systems

① $\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y)$

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \frac{g(x_0,y_0)}{f(x_0,y_0)}$$

= unique – independent of t – i.e. independent of "when" the particle passed through (x_0, y_0)

② $\frac{dx}{dt} = f(x,y,t) \qquad \frac{dy}{dt} = g(x,y,t)$

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \frac{g(x_0,y_0,t)}{f(x_0,y_0,t)}$$

not unique – if starting time changes then the particle reaches (x_0, y_0) at a different time.

Stability (of autonomous systems)

Any points at which $\vec{x}' = \vec{f}(\vec{x}) = \vec{0}$ are called <u>critical points</u>. Once the "particle" arrives at a critical point it never leaves since its velocity is zero.

In 9.1 the only critical points were $\vec{X}^\circ = \vec{0}$.

A critical point $\vec{X}^\circ$ of $\dot{\vec{X}} = \vec{f}(\vec{X})$ is <u>stable</u> if

$$\forall\, \varepsilon > 0\ \exists\ \delta > 0 \quad \text{s.t} \quad \|\vec{X}(0) - \vec{X}^\circ\| < \delta \implies \|\vec{X}(t) - \vec{X}^\circ\| < \varepsilon \quad \forall\, t \geq 0.$$

If the point is not stable it is <u>unstable</u>.

$\vec{X}^\circ$ is <u>asymptotically stable</u> if it is stable and if $\exists\ \delta > 0$ s.t.

$$\|\vec{X}(0) - \vec{X}^\circ\| < \delta \implies \lim_{t \to \infty} \vec{X} = \vec{X}^\circ$$