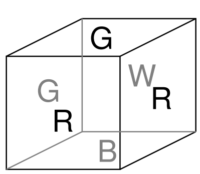
Application of Graphs: Instant Insanity

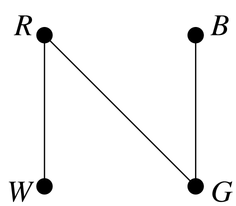
MAT230 Discrete Mathematics

Written Fall 2010, Revised Fall 2016

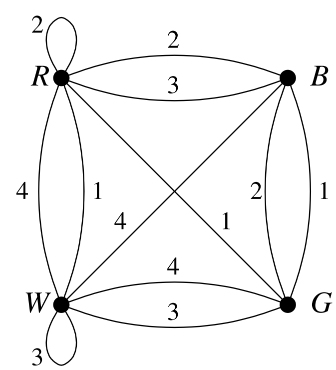
In the 1960's a puzzle game called *Instant Insanity* was popular. To play the game one needs four cubes with each face colored with one of four colors. The challenge of the game is to arrange the four cubes in a stack such that all four colors are present on each of the four sides of the stack. Solving the puzzle usually requires a lot of trial-and-error.

Graph theory provides a helpful tool to solve this puzzle, but first we must carefully lay some groundwork. We will refer to the six sides of a cube in pairs as *front* and *back*, *right* and *left*, and *top* and *bottom*. Each cube can be diagrammed using a graph that shows the relationship between the colors of opposite sides. The graph has four vertices, one four each of the four colors appearing on the cube, and edges that connect vertices corresponding to colors on opposing faces of the cube. For example, the cube shown below has a red front and white back, a red right side and a green left side, and a green top and blue bottom. The graph shows edges connecting the colors of opposite sides.





We can draw separate graphs for each cube, but it is more useful to label the edges with a number to indicate which cube the edges belong to, and draw all the graphs for all four cubes using the same set of vertices. Here is a graph describing four cubes:



Consider the two circuits

$$B - 1 - G - 3 - W - 4 - R - 2 - B,$$
$$W - 1 - R - 3 - B - 2 - G - 4 - W$$

The first starts at vertex B, traverses an edge labeled 1 to vertex G, traverses an edge labeled 3 to W, etc. until it returns to B via an edge labeled 2; the second follows a similar pattern. Both circuits use each of the four colors and edges corresponding to each of the four cubes. Note that if an edge is used in one circuit, it is not used in the other circuit. These two circuits provide a solution to the puzzle.

The first circuit tells us the colors that will be visible on the front and back of the stack of cubes and the second circuit provides similar information for the right and left sides of the stack. To start constructing the stack, we look at the first circuit and see that it includes $B-1-G$, indicating that cube 1 as blue and green on two of its opposing faces. We can therefore take the first cube and orient it so that the front face is blue and the back face is green. By examining the second circuit, we see it contains $W - 1 - R$, indicating that white and red are also on opposing faces on cube 1. Keeping the blue face in front, we can rotate the cube to get a white face on the right side and a red face on the left side. This will be the bottom cube in the stack.

It does not matter which of the remaining cubes we choose next; cube 3 appears next in both circuits (a coincidence) so we might as well use it. The first circuit contains $G-3-W$ so cube 3 will have a green face in front and a white face in back. We see $R-3-B$ in the second circuit so we spin the cube so red is on the right and blue is on the left and place the cube on the stack. Continuing, the first and second circuits contain $W-4-R$ and $G-4-W$ respectively so we next orient cube 4 so white is in front, red in the back, green on the right and white on the left. Finally, we orient cube 2 so red is in front, blue in back and on the right, and green on the left. Once the last two cubes are placed on the stack the puzzle is solved!

Even when we use graphs there will still be some trial and error; the graphs just make it easier to visualize how to construct a solution. Since the puzzle is considered solved once circuits like those above are found, the challenge is finding the right circuits. Generally we begin by drawing a graph that diagrams the colors on opposite faces of each cube and then look for a pair of circuits such that each one uses all four vertices and four edges where each edge corresponds to a different cube. We must ensure, however, that no edge appears more than once in either circuit. We may have to construct several circuit pairs before finding one that is suitable.

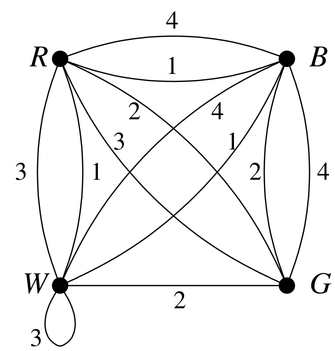
As it turns out, it is possible, and sometimes necessary, to find solutions with more than two circuits. Consider, for example, the pair of circuits

$$R-2-R, \qquad W-4-B-1-G-3-W$$

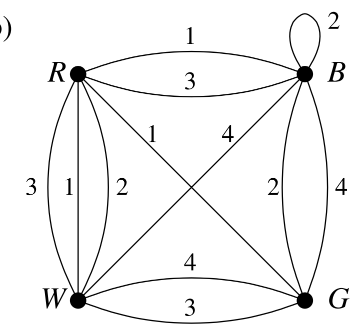
Just as in the first circuit described in the solution above, each cube is associated with exactly one edge and each color appears once (since colors at the start and end of a circuit count as one occurrence). These two circuits, like the first circuit above, describe how to orient the four cubes so the front and back colors are arranged properly. Since all four colors and all four cubes appear in the circuits, they completely determine the front and back colors. Once a circuit or circuits are found for the colors on the left and right sides of the stack the solution will be complete; one such circuit is $W-1-R-3-B-2-G-4-W$.

Exercise: Find solutions to the *Instant Insanity* puzzle given the following cube graphs:

a)



b)



References:

Discrete Mathematics, 6th Edition, by Richard Johnsonbaugh, Pearson Prentice Hall, 2005, p. 369-371.

Jaap's Puzzle Page:
`http://www.jaapsch.net/puzzles/insanity.htm`