Combinatorics

MAT230

Discrete Mathematics

Fall 2019

Outline

Basic Counting Techniques

1. How many ways are there to pick a president from a class of 256 women and 128 men?

2. How many ways are there to pick a president from 256 women and a vice president from 128 men?

Basic Counting Techniques

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   There are $256 + 128 = 384$ people, so there are 384 different ways to choose a president.

2. How many ways are there to pick a president from 256 women and a vice president from 128 men?

Basic Counting Techniques

1. How many ways are there to pick a president from a class of 256 women and 128 men?

   There are $256 + 128 = 384$ people, so there are 384 different ways to choose a president.

2. How many ways are there to pick a president from 256 women and a vice president from 128 men?

   There are 256 ways to choose a president. For each of those ways there are 128 ways to choose a vice president. Thus there are $256 \times 128 = 32,768$ ways to choose both.

Basic Counting Techniques

These problems are examples of **counting** problems. Our first two counting rules are:

1. **The Rule of Sums:** If a first task can be done n ways and a second task can be done m ways, and if these tasks cannot be done at the same time, then there are $n + m$ ways to do either task.
2. **The Rule of Products:** If a first task can be done n ways and a second task can be done m ways, and if the tasks must be done sequentially, then there are $n \cdot m$ ways to do the two tasks.

Basic Counting Techniques

The first question we encountered required the Sum Rule because we needed to choose a president and we could do that from the group of women (task 1) **or** from the group of men (task 2). The two tasks could not both be done; we could only do one or the other.

The second question required the Rule of Products because we had to pick a president from the group of women (task 1) **and** then pick a vice president from the group of men (task 2). The order of the tasks is not important, just that each one is *independent* of the other.

Basic Counting Techniques

Example

How many different bit strings having 5 bits are there?

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A bit string with 5 bits has five "slots" that can hold bits. To "create" a bit string we need to first choose the first bit, then the second bit, and so on. Thus the product rule will apply here.

number of choices:
bit string:

Basic Counting Techniques

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number of choices:	2
bit string:	

Basic Counting Techniques

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number of choices:	2
bit string:	0

Basic Counting Techniques

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number of choices:	2	2
bit string:	0	

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number of choices:	2	2	2	2	2
bit string:	0	1	1	0	1

If we find the product of all the "number of choices" values we see that there are 32 different bit strings of length 5.

Basic Counting Techniques

Example

How many two digit base 10 numbers are there?

Basic Counting Techniques

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In this case we are considering a "digit string" of length 2. This is an example that again calls for the product rule. If we allow zero as the first digit then there are 10 ways to choose the first digit. For each of these there are 10 ways to choose the second digit. So there are $10 \times 10 = 100$ different two digit base 10 numbers.

Basic Counting Techniques

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If, however, we don't want to allow zero as leading digit, then there are only 9 ways to choose the first number, but still 10 ways to choose the second number. Thus there are $9 \times 10 = 90$ different two digit base 10 numbers.

Basic Counting Techniques

Example

How many bit strings of length 8 start with 101 or 010?

Basic Counting Techniques

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This is again a product rule type problem. The best way to think of this is to think of there being 6 choices: Choose one of two prefixes to start the string and then choose each of the 5 remaining bits.

number of choices:	2	2	2	2	2	2
bit string:	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Basic Counting Techniques

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bit string:	101	0	1	1	0	1

There are $2^6 = 64$ bit strings of length 8 beginning with 101 or 010.

Basic Counting Techniques

Example

U.S. Novice class amateur radio (ham) radio callsigns have two letters, a single digit, and then three more letters. The first letter must be K, N, or W. How many different novice callsigns are there?

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As in the other examples, we have 3 ways to choose the first letter, 26 ways to choose the second. Next we need to pick a digit; there are 10 ways to do this. Finally we have to pick three more letters and have 26 choices for each. Thus we have

$$3 \times 26 \times 10 \times 26 \times 26 \times 26 = 13,709,280$$

There are 13,709,280 possible novice class callsigns.

(The real allowable number is somewhat less than this as certain three-letter combinations are disallowed).

Basic Counting Techniques

Example

How many ways can four people be arranged in a line for a photograph?

Basic Counting Techniques

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How many ways can four people be arranged in a line for a photograph?

There are 4 ways to choose a person to stand at the far left. There are 3 ways to choose the next person, then 2 ways to choose the next, and only 1 way to choose the last person to stand on the right. Thus there are

$$4 \times 3 \times 2 \times 1 = 4! = 24$$

ways to arrange four people for a photograph.

Basic Counting Techniques

Example

How many ways can four people be arranged at a card table?

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How many ways can four people be arranged at a card table?

As before, there are $4! = 24$ ways to arrange the four people. However, the arrangement ABCD is equivalent to the arrangements BCDA, CDAB, and DABC. There are four such "equivalent" arrangements.

Thus there are really only

$$\frac{4!}{4} = 3! = 6$$

different ways to arrange the four people at the card table.

Permutations

How many ways are there to choose a president, vice-president, and secretary from a group of 17 people?

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How many ways are there to choose a president, vice-president, and secretary from a group of 17 people?

There are 17 choices for president, but then only 16 choices for vice-president, and finally only 15 choices for secretary. (This assumes that no person can hold two or more offices at the same time). This means that there are

$$17 \times 16 \times 15 = 4,080$$

different outcomes in the election.

Permutations

Notice that

- The order of selection matters. The two election outcomes

President:	Alice
Vice-President:	Bob
Secretary:	Mark

President:	Bob
Vice-President:	Mark
Secretary:	Alice

  represent two different election outcomes even thought the same three people are elected.
- Repetition is **not** allowed.
- We made use of the Rule of Products.

Permutations

Definition

If n is a positive integer then n **factorial** is the product of the first n positive integers and is denoted $n!$. Additionally, we define $0!$ to be 1.

Definition

An ordered arrangement of k elements selected from a set of n elements, $0 < k \leq n$, where no two elements of the arrangement are the same, is called a **permutation of n objects taken k at a time**. The total number of such permutations is denoted by $P(n, k)$.

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$
$$= \frac{n!}{(n-k)!}$$

Combinations

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Answer: Now *order does not matter:* Amy, Jessica, and Mike are the same three winners if we list them as Mike, Jessica, and Amy. To count this, we need to compute the number of permutations **and then divide** by the number of ways the winners can be arranged: Thus, the number of possible outcomes is

$$\frac{P(20, 3)}{3!} = \frac{20 \cdot 19 \cdot 18}{6} = 1,140.$$

Combinations

The number of **combinations** of n objects "taken" k at a time is computed as

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$

There are several different notations used for counting combinations and permutations:

Permutations: $P(n,k), \quad P(n,k), \quad {}_nP_k, \quad P_k^n$

Combinations: $C(n,k), \quad C(n,k), \quad {}_nC_k, \quad C_k^n, \quad \binom{n}{k}$

The last form shown here, $\binom{n}{k}$, is often referred to as a **binomial coefficient**.

We usually read $C(n,k)$ or $\binom{n}{k}$ as "n choose k."

Combinations

Example

How many bit strings of length four have exactly two 1's and two 0's? To answer this, consider that there are four locations we need to fill with bits:

$$\_\,\_\,\_\,\_.$$

To get a bit string with two 1's we first choose two locations to place the 1's. For example:

$$\_\,1\,\_\,1.$$

There are $C(4,2) = \frac{4!}{2!(4-2)!} = \frac{4\cdot 3}{2\cdot 1} = 6$ ways to do this. Now place the 0's:

$$0\,1\,0\,1.$$

There is only $C(2,2) = 1$ way to do this. By the Rule of Products, there are $C(4,2)\cdot C(2,2) = 6\cdot 1 = 6$ bit strings of length four with exactly two 1's and two 0's.

Permutations when Repetition is Allowed

Question: An urn contains 4 numbered red balls and 2 numbered white balls. Balls are drawn sequentially and designated "first" and "second." How many samples of two balls contain two red balls?

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Answer: We need to choose 2 of the 4 red balls. Since the balls are numbered it makes a difference which two we choose. Also, note that the order the balls are drawn is significant. We need to use a permutation; in this case $P(4, 2) = 4 \cdot 3 = 12$.

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Now suppose that once a ball is selected and the color is observed, it is replaced into the urn before the next selection. How many samples contain two red balls now?

$$4 \cdot 4 = 16$$

Theorem

The number of k-permutations with repetition of a set of n objects is n^k. This is also called an k-sequence.

Combinations when Repetition is Allowed

Question: How many ways can four bills from a money drawer with six denominations be selected?

Combinations when Repetition is Allowed

Question: How many ways can four bills from a money drawer with six denominations be selected?

In answering this note the following:

- We can repeatedly choose a particular denomination
- We can think of the money drawer as list of bills and separators:

$$\$100 \mid \$50 \mid \$20 \mid \$10 \mid \$5 \mid \$1$$

- If we "select" a bill by placing a marker in that denomination's bin, then the problem at hand reduces to "How many ways can 4 markers and 5 separators be arranged?"

Combinations when Repetition is Allowed

For example, if we chose two \$50 bills, a \$10 bill, and a \$5 bill the arrangement would look like

	x x		x	x	
\$100	\$50	\$20	\$10	\$5	\$1

Considering only the 5 separators and 4 markers

$$| \; x \; x \; | \quad | \; x \; | \; x \; |$$

we see there are 9 items and they can be arranged $C(9,4)$ (or $C(9,5)$) different ways.

So, there are $C(9,4) = 126$ ways to select four bills from a money drawer with six denominations.

Combinations when Repetition is Allowed

Theorem

There are $C(n+k-1, k)$ k-combinations with repetition from the set with n elements. These are also called k-collections.

Notice that $C(n+k-1, k) = C(n-1+k, n-1)$, an alternative form sometimes used.

Example: A cookie shop has 9 varieties of cookies. How many different ways are there to select

- a dozen cookies?

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- a dozen cookies with at least one of each type?

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$$C(9+12-1, 12) = 125,970 \text{ ways.}$$

- a dozen cookies with at least one of each type? There is one way to pick one cookie of each type. This leaves 3 cookies to pick.

$$1 \cdot C(9+3-1, 3) = 165 \text{ ways.}$$

Summary of Combinatoric Formulas

Suppose there is a set with n elements from which we need to select k elements. The following table summarizes the formulas needed to compute the number of ways the selection can be made.

		Does order matter?	
		yes	no
Is repetition allowed?	yes	Ordered list n^k	Unordered list $C(n+k-1,k)$
	no	Permutation $P(n,k)$	Set $C(n,k)$

The Binomial Theorem

The expression $(x + y)$ is called a **binomial** since it consists of two terms.

Theorem (Binomial Theorem)

If $n \geq 0$, then, for expressions x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

Example

$$(5+2a)^3 = \binom{3}{0} 5^3 (2a)^0 + \binom{3}{1} 5^2 (2a)^1 + \binom{3}{2} 5^1 (2a)^2 + \binom{3}{3} 5^0 (2a)^3$$
$$= 1 \cdot 125 \cdot 1 + 3 \cdot 25 \cdot 2a + 3 \cdot 5 \cdot 4a^2 + 1 \cdot 1 \cdot 8a^3$$
$$= 125 + 150a + 60a^2 + 8a^3$$

Binomial Coefficients and Pascal's Triangle

One way to generate the binomial coefficients is with **Pascal's Triangle**.

Notice how each entry in the triangle is the sum of the two elements above it.

This relationship is described by **Pascal's Identity**:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

```
                        1
                     1     1
                  1     2     1
               1     3     3     1
            1     4     6     4     1
         1     5    10    10     5     1
      1     6    15    20    15     6     1
                                \   /
   1     7    21    35    35    21     7     1
```

Pascal's Identity

Theorem

Let n and k be positive integers with $n > k$. Then

$$C(n,k) = C(n-1,k-1) + C(n-1,k).$$

Proof.

Let A be a set with $|A| = n$. Pick any one element of A and label it x, leaving the remaining $n-1$ elements of A unlabeled. Now choose k elements of A to form a subset S. There are two cases:

1. $x \in S$: There are $C(n-1,k-1)$ ways the remaining $k-1$ elements of S could have been chosen from the $n-1$ unlabeled elements of A.
2. $x \notin S$: All k elements of S must have been chosen from the $n-1$ unlabeled elements of A. There are $C(n-1,k)$ ways to do this.

Since one or the the other of these cases must occur, their sum must equal $C(n,k)$, the number of ways to form a subset with k elements from a set with n elements. Thus $C(n,k) = C(n-1,k-1) + C(n-1,k)$. □

Partitions

Recall that a **partition** of a set A is a collection of disjoint, nonempty subsets of A that have A as their union. Our current text focuses this a bit more, saying that a partition of a set A is a *set* of one or more nonempty subsets of A that have A as their union.

Example

Suppose $A = \{a, b, c\}$. The following are all partitions of A:

- $\{\{a\}, \{b\}, \{c\}\}$
- $\{\{a, b\}, \{c\}\}$
- $\{\{a, b, c\}\}$.

Basic Law of Addition

Definition

The **Basic Law of Addition** says that if A is a finite set and if $\{A_1, A_2, \ldots, A_n\}$ is a partition of A, then

$$
\begin{aligned}
|A| &= \sum_{k=1}^{n} |A_k| \\
&= |A_1| + |A_2| + \cdots + |A_n| \\
&= |A_1 \cup A_2 \cup \cdots \cup A_n|.
\end{aligned}
$$

Example

Think back to when you were learning to count and do basic math in elementary school. The numbers 3 and 5 might have been represented by a group of three blocks and a group of five blocks. Adding $3 + 5$ meant counting the number of blocks in both groups.

Inclusion-Exclusion

Note that

$$|A_1 \cup A_2| = |A_1| + |A_2|$$

when A_1 and A_2 are disjoint. What if they are not disjoint? How can we compute $|A_1 \cup A_2|$?

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Adding $|A_1|$ and $|A_2|$ won't work because we count elements that are common to both sets twice. However, since we count them *exactly* twice, we can correct for this by subtracting the number of elements common to both sets.

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The **Principle of Inclusion-Exclusion** for any two finite sets A_1 and A_2 says

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Inclusion-Exclusion

Example

Consider the set of students in our class.

1. Let all students who like Dunkin Donuts coffee belong to set D. $|D| = ?$
2. Let all students who like Starbucks coffee belong to set S. $|S| = ?$
3. How many of the students in our class like both Dunkin Donuts coffee and Starbucks coffee?
4. How many students students like Dunkin Donuts coffee or Starbucks coffee?

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