Probability

MAT230

Discrete Mathematics

Fall 2019

Outline

1. Discrete Probability
2. Sum and Product Rules for Probability
3. Expected Value

Introduction to Probability

Example

Suppose we roll a pair of dice and record the sum of the face-up numbers. what is the likelihood the sum is 8?

- This is a question about **probability**.
- Before we begin our study, we first need to define some terms.

Definitions

- The **probability of an event** is a number which expresses the long-run likelihood that the event will occur.
- An **experiment** is an activity with an observable outcome.
- Each repetition of an experiment is called a **trial**.
- The result of an experiment is called an **outcome**.
- The set of all possible outcomes is called the **sample space**.

For example:

- Rolling a pair of dice is an **experiment**. We can represent the **outcome** with an ordered pair, such as $(2,3)$, to indicate the number shown on each die.
- We use an ordered pair rather than a set because we need to count $(2,3)$ and $(3,2)$ as two separate outcomes. This means the dice are **distinguishable** from one another.

The Sample Space

When a pair of dice is rolled, the **sample space** S is given by

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

so that $|S| = 36$.

We can display S in a table:

		Die 2					
		1	2	3	4	5	6
Die 1	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The Sample Space

In our original example we wanted to find the likelihood the sum of face-up numbers is 8 when a pair of dice are rolled.

If E is the event "the sum of face-up numbers on the two dice is 8," we can count the ordered pairs that sum to 8 to find $|E|$.

		1	2	3	4	5	6
Die 1	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

There are five ordered pairs that sum to 8: (2,6), (3,5), (4,4), (5,3), and (6,2) so $|E| = 5$.

Probability of Equally Likely Outcomes

Definition

Given an experiment with a sample space S of *equally likely* outcomes and an event E, the **probability of the event** is computed

$$P(E) = \frac{|E|}{|S|}.$$

In our dice-rolling example we note that each outcome in S is equally likely (assuming the dice are not loaded) so we can compute

$$P(E) = \frac{5}{36} = 0.13\overline{8}.$$

This means that when many trials are conducted, we would expect the sum of face-up numbers will be 8 just under 14% of the time.

Example 2

Example

Roll a pair of dice. What is the probability that they show the same number?

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The sample space is the same as before so $|S| = 36$. The event E is "both dice show the same number." We can proceed two different ways:

1. Count the ordered pairs in S of the form (k, k). There are six of them: $(1,1), (2,2), \ldots, (6,6)$ so $|E| = 6$.
2. Use combinatorics:

$$\begin{aligned} |E| &= (\text{6 possibilities on first die}) \times (\text{1 way to match the first die}) \\ &= 6. \end{aligned}$$

Either way, we can compute $P(E) = 6/36 = 1/6$. If the experiment was repeated many times, we'd expect a roll producing a pair would occur $1/6 \approx 16.7\%$ of the time.

Example 3

Example

What is the probability that when a fair coin is tossed ten times it comes up heads exactly 5 times?

Example 3

Example

What is the probability that when a fair coin is tossed ten times it comes up heads exactly 5 times?

- The sample space S is the set of all outcomes of ten tosses and E is the subset of S corresponding to there being 5 heads and 5 tails in the ten tosses.
- Elements in S could be represented as strings consisting of H and T. For example "HTTHHTTHTH" is an element of both S and E.

$$|S| = 2^{10} = 1024, \qquad |E| = C(10,5) = 252$$

$$P(E) = \frac{252}{1024} \approx 0.2461$$

We therefore expect exactly five heads nearly 25% of the time.

Example 4

Example

A bin containing 24 apples has 6 apples with worms, the remainder are worm-free. If 5 apples are selected without examination, what is the probability that

1. all are worm-free?
2. at most one has a worm?
3. all have worms?

Example 4

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A bin containing 24 apples has 6 apples with worms, the remainder are worm-free. If 5 apples are selected without examination, what is the probability that

1. all are worm-free?
2. at most one has a worm?
3. all have worms?

Solution: Let S be the set of all outcomes when 5 apples are chosen from a group of 24 so $|S| = C(24,5) = 42{,}504$.

1. Let E be the event "all chosen apples are worm-free." In this case we're only choosing from the $24 - 6 = 18$ worm-free apples so $|E| = C(18,5)$.

$$P(E) = \frac{C(18,5)}{C(24,5)} = \frac{8{,}568}{42{,}504} = \frac{51}{253} \approx 0.2016.$$

Example 4 (Continued)

Example (Continued)

2. Let E be "at most one of the 5 chosen apples has a worm." Then

$$\begin{aligned}|E| &= (\#\text{ ways to choose only worm-free apples}) \\ &\quad + (\#\text{ ways to choose 1 wormy apple and 4 worm-free apples}) \\ &= C(18,5) + C(6,1) \cdot C(18,4) = 8{,}568 + 18{,}360 = 26{,}928\end{aligned}$$

$$P(E) = \frac{26{,}928}{42{,}504} = \frac{102}{161} \approx 0.6335.$$

3. Let E be "all 5 apples have worms." Then

$$|E| = C(6,5) = 6$$

$$P(E) = \frac{6}{42{,}504} = \frac{1}{7{,}084} \approx 0.0001412.$$

Complementary Events

Suppose S is the sample space of an experiment and E is the set of outcomes comprising a certain event. We say the **complement** of E is the event that E does not occur and note that $\overline{E} = S - E$. Then

$$P(\overline{E}) = \frac{|\overline{E}|}{|S|} = \frac{|S - E|}{|S|} = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|}$$
$$= 1 - P(E).$$

- Given an event E, the event E either occurs or it does not occur; *one or the other must happen* so $P(E) + P(\overline{E}) = 1$.
- Sometimes it may be easier to compute the probability of an event by first computing the probability the event does not occur and then subtract this value from 1.

Example 5

Example

Suppose five people each pick a single digit number from $\{0, 1, \ldots, 9\}$. What is the probability that

1. exactly two people pick the same number?
2. at least two people pick the same number?

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We first note that an outcome for this experiment can be represented as a string of 5 digits, so S consists of digit strings of length 5 and $|S| = 10^5$.

(Continued)

Example 5 (Continued)

Example (Continued)

1. Let E be the event "exactly two people picked the same number." Then

$$\begin{aligned} |E| &= (\#\text{ ways to choose two locations for repeated digit}) \\ &\quad \cdot (\#\text{ ways to choose four distinct digits}) \\ &= C(5,2) \cdot P(10,4) = 50{,}400. \end{aligned}$$

The probability that exactly two people pick the same number is given by

$$P(E) = \frac{50{,}400}{10^5} = \frac{504}{1000} = 0.504.$$

(Continued)

Example 5 (Continued)

Example (Continued)

2. Next, let E be the event "at least two people pick the same number." In this case $\overline{E}$ would be "no two people picked the same number" and $|\overline{E}| = P(10, 5) = 30{,}240$. Then

$$\begin{aligned} P(E) &= 1 - P(\overline{E}) \\ &= 1 - \frac{30{,}240}{10^5} \\ &= 1 - 0.3024 \\ &= 0.6976 \end{aligned}$$

is the probability that at least two people pick the same number.

Example 5 (Continued)

Example (Continued)

It's worth pointing out just how much simplier it is to use the complement in our calculuation. If E is "at least two people pick the same number,"

$$
\begin{aligned}
|E| &= C(5,2)\cdot P(10,4) && \text{exactly 2 people pick same}\\
&\quad + C(5,2)\cdot C(3,2)\cdot P(10,3)/2 && \text{exactly 2 pair pick same}\\
&\quad + C(5,3)\cdot P(10,3) && \text{exactly 3 people pick same}\\
&\quad + C(5,3)\cdot P(10,2) && \text{2 people and 3 people pick same}\\
&\quad + C(5,4)\cdot P(10,2) && \text{exactly 4 people pick same}\\
&\quad + C(5,5)\cdot P(10,1) && \text{all 5 people pick same}
\end{aligned}
$$

So

$$|E| = 50{,}400 + 10{,}800 + 7{,}200 + 900 + 450 + 10 = 69{,}760$$

$$P(E) = \frac{69{,}760}{10^5} = 0.6976$$

Disjoint Events

Definition

Two events are **disjoint** if they cannot occur simultaneously.

Example

The result of tossing a coin cannot be both "heads" and "tails," so

- E_1 = "comes up heads" and
- E_2 = "comes up tails"

are disjoint.

Disjoint Events

Example

Toss a pair of dice. Suppose

1. E_1 is "first die shows an even number,"
2. E_2 is "sum of numbers shown is 4," and
3. E_3 is "both dice show odd numbers."

Which of the following pairs of events are disjoint?

- E_1 and E_2:
- E_1 and E_3:
- E_2 and E_3:

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- E_1 and E_3: disjoint
- E_2 and E_3: not disjoint

Sum Rule

Theorem (Sum Rule)

If E_1 and E_2 are disjoint events in an experiment, the probability of E_1 or E_2 is

$$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$$

Note that $P(E_1 \text{ or } E_2)$ could also be written as $P(E_1 \cup E_2)$.

Example

From our last example:

$$\begin{aligned} P(\text{first die shows an even number or both dice show odd numbers}) \\ &= \frac{3}{6} + \frac{9}{36} \\ &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 0.75. \end{aligned}$$

General Sum Rule

What is the probability that a card selected at random from a deck of 52 cards is a spade or an ace?

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Let E_1 be "card is spade" and E_2 be "card is an ace." Then

$$P(E_1) = \frac{C(13,1)}{C(52,1)} = \frac{13}{52} = \frac{1}{4}, \qquad P(E_2) = \frac{C(4,1)}{C(52,1)} = \frac{4}{52} = \frac{1}{13}.$$

We cannot merely add $P(E_1)$ and $P(E_2)$ since E_1 and E_2 are not disjoint – the card could be the ace of spades.

How can we proceed?

General Sum Rule

Given an experiment with a sample space S and two events E_1 and E_2, both with equally likely outcomes,

$$\begin{aligned} P(E_1 \text{ or } E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} \\ &= P(E_1) + P(E_2) - P(E_1 \text{ and } E_2). \end{aligned}$$

This generalizes as

Theorem (General Sum Rule)

If E_1 and E_2 are any events in an experiment, the probability of E_1 or E_2 is

$$\begin{aligned} P(E_1 \text{ or } E_2) &= P(E_1) + P(E_2) - P(E_1 \text{ and } E_2), \text{ or} \\ P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2). \end{aligned}$$

General Sum Rule

Returning to our problem: What is the probability that a card selected at random from a deck of 52 cards is a spade or an ace?

Let E_1 be "card is spade" and E_2 be "card is an ace." Then

$$P(E_1) = \frac{C(13,1)}{C(52,1)} = \frac{13}{52} = \frac{1}{4}, \qquad P(E_2) = \frac{C(4,1)}{C(52,1)} = \frac{4}{52} = \frac{1}{13},$$

$$P(E_1 \cap E_2) = \frac{C(1,1)}{C(52,1)} = \frac{1}{52}.$$

so

$$\begin{aligned} P(E_1 \text{ or } E_2) &= P(E_1) + P(E_2) - P(E_1 \text{ and } E_2) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{16}{52} \approx 0.3077 \end{aligned}$$

Independence

Definition

Two events are **independent** if the occurrence of one event is not influenced by the occurrence or non-occurrence of the other event.

Determine if E_1 and E_2 are independent events in each of the following two scenarios:

A single die is tossed twice. Let E_1 be "first toss is a 5" and E_2 be "second toss is even."

Two cards are chosen from a 52-card deck. Let E_1 be "at least one card is an ace" and E_2 be "at least one card is a king."

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Yes, these are independent events. The outcome of E_1 has no impact on E_2 or vice-versa (assuming the die is fair).

Two cards are chosen from a 52-card deck. Let E_1 be "at least one card is an ace" and E_2 be "at least one card is a king."

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Two cards are chosen from a 52-card deck. Let E_1 be "at least one card is an ace" and E_2 be "at least one card is a king."

No. If we know that one card is an ace, and therefore not a king, it reduces the probability that one of the cards is a king.

Product Rule

Theorem

If E_1 and E_2 are independent events in a given experiment, the probability that both E_1 and E_2 occur is

$$P(E_1 \text{ and } E_2) = P(E_1) \cdot P(E_2).$$

Example

A single die is tossed twice. Let E_1 be "first toss is a 5" and E_2 be "second toss is even." The probability that both E_1 and E_2 occur is

$$P(E_1 \text{ and } E_2) = \frac{1}{6} \cdot \frac{3}{6} = \frac{1}{12}.$$

How could we proceed if the E_1 and E_2 are dependent?

Conditional Probability and the General Product Rule

Definition (Conditional Probability)

Given events E_1 and E_2 for some experiment, the **conditional probability of** E_1 **given** E_2, denoted $P(E_2|E_1)$, is the probability that E_2 occurs given that E_1 occurs.

Theorem (General Product Rule)

If E_1 and E_2 are any events in a given experiment, then

$$P(E_1 \textit{ and } E_2) = P(E_1) \cdot P(E_2|E_1), \quad \textit{or}$$

$$P(E_2|E_1) = \frac{P(E_1 \textit{ and } E_2)}{P(E_1)} = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

Conditional Probability and the General Product Rule

Example

Example: A coin is tossed three times. What is the probability that it comes up heads all three times given that it comes up heads the first time?

Conditional Probability and the General Product Rule

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Solution:

$$E_1 = \text{“heads occurs first time,”} \qquad E_2 = \text{“heads occurs three times”}$$

$$|S| = 2^3 = 8, \qquad |E_1| = 4, \qquad |E_2| = 1, \qquad |E_1 \cap E_2| = 1$$

$$P(E_1 \cap E_2) = \frac{1}{8}, \qquad P(E_1) = \frac{4}{8} = \frac{1}{2}$$

$$P(E_2|E_1) = \frac{1/8}{1/2} = \frac{1}{4} = 0.25.$$

This answer makes sense. If the first toss is known to be an head, we just need the next two out of two tosses to be heads, and the probability of that is $1/4$.

General Product Rule

Example

Two cards are chosen from a 52-card deck. Let E_1 be "at least one card is an ace" and E_2 be "at least one card is a king." Note that $P(E_1) = P(E_2)$.

$$\begin{aligned} P(E_1) &= P(\text{at least one card is an ace}) \\ &= 1 - P(\text{neither card is an ace}) \\ &= 1 - \frac{C(48,2)}{C(52,2)} = 1 - \frac{1128}{1326} = \frac{33}{221} \approx 0.1493 \end{aligned}$$

$$P(E_1 \text{ and } E_2) = \frac{C(4,1)C(4,1)}{C(52,2)} = \frac{16}{1326} = \frac{8}{663} \approx 0.01207$$

$$P(E_2|E_1) = \frac{8/663}{33/221} = \frac{8}{99} \approx 0.0808$$

Bernoulli Trials

Some experiments have only two possible outcomes, e.g. tossing a coin; such experiments are called **Bernoulli Trials**. We can easily compute the probability that one of these outcomes will occur a particular number of times in a sequence of trials.

- Suppose an experiment has only two possible outcomes. Let p be the probability of "success" (the stipulated event *does* occur) and q be the probability of "failure" (the event does not occur).
- Notice that $q = 1 - p$.
- If there are to be n trials of the experiment then the probability of k successes in the n trials is given by

$$P_B = C(n,k)p^k q^{n-k}.$$

Bernoulli Trials

Example

What is the probability that when a fair coin is tossed ten times it comes up heads exactly 5 times?

Bernoulli Trials

Example

What is the probability that when a fair coin is tossed ten times it comes up heads exactly 5 times?

Solution: In this case $p = q = 0.5$ so

$$\begin{aligned} P(\text{heads exactly 5 times in 10 tosses}) &= C(10,5)(0.5)^5(0.5)^5 \\ &\approx 0.2461 \end{aligned}$$

We therefore expect exactly five heads nearly 25% of the time.

Bernoulli Trials

Example

Suppose the probability a certain baseball player will get a hit during each at-bat is $1/3$. What is the probability the player gets exactly one hit in four at-bats?

Bernoulli Trials

Example

Suppose the probability a certain baseball player will get a hit during each at-bat is $1/3$. What is the probability the player gets exactly one hit in four at-bats?

Solution: We take $p = 1/3$ so $q = 2/3$. Then

$$\begin{aligned} P(1 \text{ hit in } 3 \text{ at-bats}) &= C(4,1)\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 \\ &= 4 \cdot \frac{8}{81} \\ &\approx 0.3951 \end{aligned}$$

Introduction to Expected Values

Consider the experiment:

A fair coin is tossed five times and the outcome is recorded.

If this experiment was repeated many times, what is the average number of heads that would show up in each experiment?

We can think of this as a *weighted average* using probabilities as weights. The probabilities are

$$P(0) = \frac{C(5,0)}{2^5} = \frac{1}{32}, \quad P(1) = \frac{C(5,1)}{2^5} = \frac{5}{32}, \quad P(2) = \frac{C(5,2)}{2^5} = \frac{10}{32},$$

$$P(3) = \frac{C(5,3)}{2^5} = \frac{10}{32}, \quad P(4) = \frac{C(5,4)}{2^5} = \frac{5}{32}, \quad P(5) = \frac{C(5,5)}{2^5} = \frac{1}{32}.$$

Using these as weights, the average number of heads is computed

$$\frac{1}{32} \cdot 0 + \frac{5}{32} \cdot 1 + \frac{10}{32} \cdot 2 + \frac{10}{32} \cdot 3 + \frac{5}{32} \cdot 4 + \frac{1}{32} \cdot 5 = \frac{80}{32} = 2.5 \text{ heads}$$

Random Variables

Definition

Suppose the set S is the sample space of an experiment. A **random variable** is a function $X : S \to \mathbb{R}$ from the sample space to the real numbers.

Note that a random variable is not random and is not a variable!

For example, we can define a random variable X for our "toss a coin five times" experiment so that X returns the number of heads recorded in 5 tosses.

- Domain is the set of strings of length five consisting of T and H.
- Range is $\{0, 1, 2, 3, 4, 5\}$.

Some examples:

$$X(\text{HTTHT}) = 2, \qquad X(\text{TTTHT}) = 1, \qquad X(\text{HHHHH}) = 5.$$

Expected Value

We use the notation $P(X = x_o)$ to mean the "probability that X is has the value x_o." In our coin-tossing experiment, $P(X = 2)$ is the probability that heads occurs exactly twice in five tosses of a fair coin.

Definition

For a given probability experiment, let X be a random variable whose possible values are from the set of numbers $\{x_1, \ldots, x_n\}$. Then the **expected value of X**, denoted $E[X]$, is the sum

$$E[X] = x_1 \cdot P(X = x_1) + x_2 \cdot P(X = x_2) + \cdots + x_n \cdot P(X = x_n)$$
$$= \sum_{i=1}^{n} x_i P(X = x_i)$$

Example 1

Suppose a pair of dice is tossed. What is the expected value of the sum of numbers shown on the two dice?

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Suppose a pair of dice is tossed. What is the expected value of the sum of numbers shown on the two dice?

Let X map ordered pairs of numbers shown on the dice to the sum of the numbers, i.e., $X(m,n) = m + n$ where $m, n \in \{1,2,3,4,5,6\}$. Then

$$\begin{array}{ccc} & P(X=2) = 1/36, & P(X=3) = 2/36, \\ P(X=4) = 3/36, & P(X=5) = 4/36, & P(X=6) = 5/36, \\ P(X=7) = 6/36, & P(X=8) = 5/36, & P(X=9) = 4/36, \\ P(X=10) = 3/36, & P(X=11) = 2/36, & P(X=12) = 1/36. \end{array}$$

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Let X map ordered pairs of numbers shown on the dice to the sum of the numbers, i.e., $X(m,n) = m + n$ where $m, n \in \{1,2,3,4,5,6\}$. Then

$$\begin{array}{ccc} & P(X=2)=1/36, & P(X=3)=2/36, \\ P(X=4)=3/36, & P(X=5)=4/36, & P(X=6)=5/36, \\ P(X=7)=6/36, & P(X=8)=5/36, & P(X=9)=4/36, \\ P(X=10)=3/36, & P(X=11)=2/36, & P(X=12)=1/36. \end{array}$$

The expected value of the sum is

$$\begin{aligned} E[X] &= 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{4}{36} + 6\cdot\frac{5}{36} + 7\cdot\frac{6}{36} \\ &\quad + 8\cdot\frac{5}{36} + 9\cdot\frac{4}{36} + 10\cdot\frac{3}{36} + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36} \\ &= \frac{252}{36} = 7 \text{ (average sum over many repetitions).} \end{aligned}$$

Example 2

The Massachusetts State Lottery has a game called *Mass Cash* that involves trying to match 5 numbers chosen at random from 35.

- Matching all 5 numbers will win \$100,000
- Matching any 4 numbers will win \$250
- Matching any 3 numbers will win \$10

It costs \$1 to play the game.

1. What is the expected value of your winnings?
2. How much should you expect to earn?

The first question is asking: "If you play this game over and over, how much money would you expect to gain or lose?"

Example 2

Let X be the number of matching numbers. Then

$$P(X=5) = \frac{C(5,5)}{C(35,5)} = \frac{1}{324{,}632} \approx 0.000003080$$
$$P(X=4) = \frac{C(5,4)\cdot C(30,1)}{C(35,5)} = \frac{150}{324{,}632} \approx 0.0004621$$
$$P(X=3) = \frac{C(5,3)\cdot C(30,2)}{C(35,5)} = \frac{4350}{324{,}632} \approx 0.0133998$$

and

$$\begin{aligned} E[X] &= 100{,}000 \cdot P(X=5) + 250 \cdot P(X=4) + 10 \cdot P(X=3) \\ &\approx 0.308041 + 0.115515 + 0.133998 \approx 0.5576 \end{aligned}$$

This means that our expected winnings are about \$0.56 each time we play the game. However, since we pay \$1 to play, we expect to lose about \$0.44 each time we play.

Expected Value in Trials of Independent Events

Theorem

Suppose an experiment has n independent trials each with probability of success p. If X is the number of successful trials then

$$E[X] = np$$

In particular, this means that computing the expected value of success in a Bernoulli trial experiment requires only the product np.

Example

Suppose the probability a family has a baby girl is 0.51. What is the expected number of girls in a family with five children?

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Example

Suppose the probability a family has a baby girl is 0.51. What is the expected number of girls in a family with five children?

Expected number of girls is $5 \cdot 0.51 = 2.55$.