Cardinality of Sets

MAT231

Transition to Higher Mathematics

Fall 2014

Outline

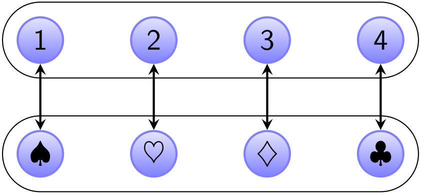
Sets with Equal Cardinality

Definition

Two sets A and B have the **same cardinality**, written $|A| = |B|$, if there exists a bijective function $f : A \to B$. If no such bijective function exists, then the sets have **unequal cardinalities**, that is, $|A| \neq |B|$.

Another way to say this is that $|A| = |B|$ if there is a one-to-one correspondence between the elements of A and the elements of B.

For example, to show that the set $A = \{1, 2, 3, 4\}$ and the set $B = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ have the same cardinality it is sufficient to construct a bijective function between them.



Sets with Equal Cardinality

Consider the following:

- This definition does not involve the number of elements in the sets.
- It works equally well for finite and infinite sets.
- Any bijection between the sets is sufficient.

Does $|\mathbb{N}| = |\mathbb{Z}|$?

True or false: $\mathbb{Z}$ is larger than $\mathbb{N}$.

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This is just the sort of ambiguity we want to avoid, so we appeal to the definition of "same cardinality." The answer to our question boils down to "Can we find a bijection between $\mathbb{N}$ and $\mathbb{Z}$?"

Does $|\mathbb{N}| = |\mathbb{Z}|$?

A first attempt at constructing a bijection between $\mathbb{N}$ and $\mathbb{Z}$ might consist of pairing $\mathbb{N}$ with the positive integers:

$\mathbb{N}$	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\mathbb{Z}$	1	2	3	4	5	6	7	8	9	10	11	$\cdots$

This starts well, but soon we realize that we'll never be able to start listing the negative integers or zero.

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An approach that does work is to list $\mathbb{Z}$ in order of magnitude.

$\mathbb{N}$	1	2	3	4	5	6	7	8	9	10	11	12	13	⋯
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	-5	5	-6	6	⋯

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This bijection between $\mathbb{N}$ and $\mathbb{Z}$ could be written as

$$f(x) = \begin{cases} (x-1)/2 & \text{when } x \text{ is odd} \\ -x/2 & \text{when } x \text{ is even} \end{cases}$$

or

$$f(x) = (-1)^{(n-1)} \lfloor x/2 \rfloor.$$

Does $|\mathbb{N}| = |\mathbb{Z}|$?

This is an interesting fact:

There are the same number of natural numbers as there are integers.

Said another way,

$$|\mathbb{N}| = |\mathbb{Z}|.$$

Does this mean that all infinite sets have the same cardinality? For example, does $|\mathbb{N}|$ equal $|\mathbb{Q}|$?

Does $|\mathbb{N}| = |\mathbb{Q}|$?

It turns out the answer is "yes, $|\mathbb{N}| = |\mathbb{Q}|$." We just need to find a bijection to show it.

For the sake of simplicity, we'll illustrate this for the positive rationals; i.e., we'll show $|\mathbb{N}| = |\mathbb{Q}^+|$. The text shows the general case and it is an obvious extension of what we see here.

Does $|\mathbb{N}| = |\mathbb{Q}^+|$?

We begin by creating a table in which each column contains all the rational numbers in simplest form with the same numerator.

1	2	3	4	5	6	7	8	$\cdots$
$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$	$\cdots$
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{2}$	$\frac{6}{5}$	$\frac{7}{2}$	$\frac{8}{3}$	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We know this table contains all the positive rationals because, no matter which positive rational we think of, it will eventually appear in the list.

Does $|\mathbb{N}| = |\mathbb{Q}^+|$?

How shall we construct a bijection between $\mathbb{N}$ and $\mathbb{Q}^+$? We can't just list $\mathbb{Q}^+$ row by row since we'll never finish with the first row.

Instead, we list in ever-increasing backward L-shaped regions, each of which has finite length.

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Does $|\mathbb{N}| = |\mathbb{Q}^+|$?

Since we've found a bijection between $\mathbb{N}$ and $\mathbb{Q}^+$ we know that they have the same cardinality.

The only change necessary to handle $\mathbb{Q}$ is to also list 0 and the negative rationals. Typically zero is listed first then the remaining rationals in positive-negative pairs.

Thus, we find that

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$$

even though there are numbers in $\mathbb{Q}$ that are not in $\mathbb{Z}$, and numbers in $\mathbb{Z}$ that are not in $\mathbb{N}$.

Countably Infinite Sets

Definition

Suppose A is a set. Then A is **countably infinite** if $|\mathbb{N}| = |A|$. The set A is **uncountable** if A is infinite and $|\mathbb{N}| \neq |A|$.

What sets are uncountable? Why can't we take any infinite set and do with it what we did with $\mathbb{Q}$ to "count" it?

$\mathbb{R}$ is Uncountable

It turns out that there are so many more reals than there are rationals that $\mathbb{R}$ is an uncountable set. Late in the 19$^{\text{th}}$ century this was proved by Georg Cantor. It is a proof by contradiction.

Proposition

The set $\mathbb{R}$ is uncountable.

Proof.

Assume, for the sake of contradiction, that $\mathbb{R}$ is countable so that there is a bijection $f : \mathbb{N} \to \mathbb{R}$. Imagine such a function represented in a table. Every natural number would be listed, as well as every real number. The next slide shows only one representative example for f, but the reasoning holds for any bijection between $\mathbb{N}$ and $\mathbb{R}$.

(continued next slide)

$\mathbb{R}$ is Uncountable

Proof (Continued).

n	$f(n)$											
1	0	.	2	1	5	0	8	9	2	0	1	...
2	9	.	5	7	3	4	3	5	8	3	4	...
3	3	.	0	0	0	3	1	4	4	9	9	...
4	1	.	9	7	5	8	2	6	5	2	9	...
5	-3	.	8	3	5	9	6	9	7	3	2	...
6	0	.	0	4	3	8	0	4	4	9	3	...
7	12	.	2	9	9	0	0	8	4	1	8	...
8	7	.	1	2	7	3	3	2	5	1	4	...
$\vdots$		$\vdots$										

(continued next slide)

$\mathbb{R}$ is Uncountable

Proof (Continued).

n	$f(n)$											
1	0	.	2	1	5	0	8	9	2	0	1	...
2	9	.	5	7	3	4	3	5	8	3	4	...
3	3	.	0	0	0	3	1	4	4	9	9	...
4	1	.	9	7	5	8	2	6	5	2	9	...
5	-3	.	8	3	5	9	6	9	7	3	2	...
6	0	.	0	4	3	8	0	4	4	9	3	...
7	12	.	2	9	9	0	0	8	4	1	8	...
8	7	.	1	2	7	3	3	2	5	1	4	...
⋮		⋮										

Construct a new number between 0 and 1 that differs from $f(n)$ in the n^{th} decimal place: Choose 0 if the digit is odd and 1 if the digit is even. In the case here we would have $0.101111100\ldots$

(continued next slide)

$\mathbb{R}$ is Uncountable

Proof (Continued).

We've just shown that we can construct a number that is not in the list of all real numbers. Since this is a contradiction, we conclude that we must not able able to find a bijection from $\mathbb{N}$ to $\mathbb{R}$. Therefore, $\mathbb{R}$ is uncountable. $\square$