Functions

MAT231

Transition to Higher Mathematics

Fall 2014

Outline

Functions

Definition

Suppose A and B are sets. A **function** f from A to B (denoted as $f : A \to B$) is a relation $f \subseteq A \times B$ from A to B, satisfying the property that for each $a \in A$ the relation f contains exactly one ordered pair of the form (a, b). The statement $(a, b) \in f$ is abbreviated $f(a) = b$.

- This definition emphases functions as sets
- Notice that functions are a special case of relations
- Notice that the sets A and B are required to specify a function

A shorter form of this definition might be

Definition

Suppose A and B are sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A.

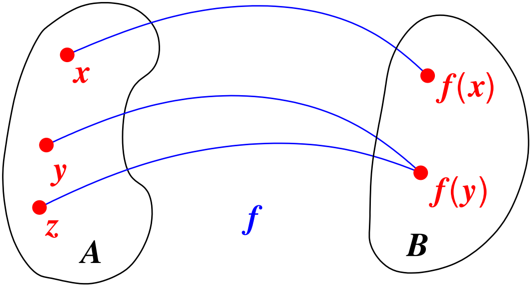
Domain and Codomain

Definition

For the function $f : A \to B$, the set A is called the **domain** of f and the set B is called the **codomain** of f. The **range** of f is the set $\{f(a) : a \in A\}$.

Note that the range of f is a subset of the codomain of f.

We can diagram $f : A \to B$ as



Function Presentation

Functions can be presented in many different forms. Some examples:

- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$
-

x	$f(x)$
a	Gödel
b	Escher
c	Bach

- $f : \mathbb{R} \to \mathbb{Z}$, $f(a) = 1$.

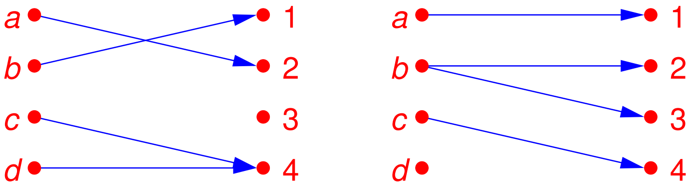
Whether explicitly or implicitly stated, the specification of a function must include its domain and codomain. The second example describes the function $f : A \to B$ with $A = \{a, b, c\}$ and $B = \{\text{Gödel}, \text{Ecsher}, \text{Bach}\}$.

Functions

Given

$$A = \{a, b, c, d\}, \qquad B = \{1, 2, 3, 4\},$$

which of the following are functions from A to B?



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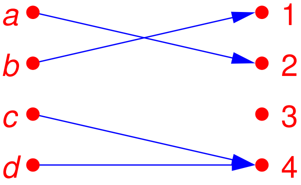

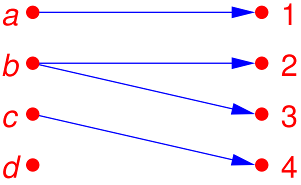

- The diagram on the left shows a function. Notice that the range is a strict subset of the codomain.

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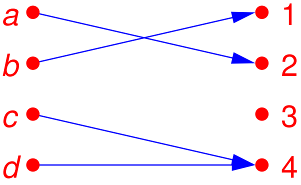

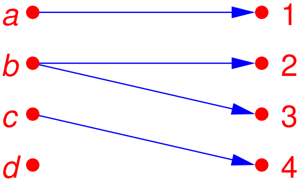

- The diagram on the left shows a function. Notice that the range is a strict subset of the codomain.
- The diagram on the right does not show a function.
  1. The element b maps to two different elements in the codomain, and
  2. the element d does not map to any element in the codomain.

Functions

Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$?

- $f = \{(x, x) : x \in \mathbb{R}\}$.
- $g = \{(x, x^2) : x \in \mathbb{R}\}$.
- $h = \{(x^2, x) : x \in \mathbb{R}\}$.
- $j = \{(x, x^3) : x \in \mathbb{R}\}$.
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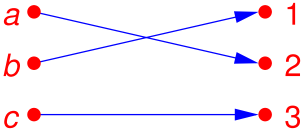
- $m = \{(x, y) : x, y \in \mathbb{R}\}$. **Not** a function; $(1, 1)$ and $(1, 2)$ both in m.

Injective Functions

Definition

A function $f : A \to B$ is **injective** (or **one-to-one**) if for every $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$. Such a function is called an **injection**.

Suppose $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.



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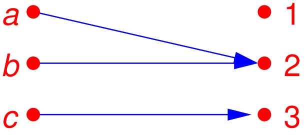

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Injective Functions

To show a function $f : A \to B$ is injective, can do one of the following

1. Assume that $f(x) = f(y)$ for some $x, y \in A$ and conclude that $x = y$.
2. Assume that $x, y \in A$ but $x \neq y$ and conclude that $f(x) \neq f(y)$.

Usually the first approach is the most natural.

Show that the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 3x + 8$ is injective.

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Proof.

Suppose $x, y \in \mathbb{Z}$ such that $f(x) = f(y)$. Then

$$\begin{aligned} 3x + 8 &= 3y + 8 \\ 3x &= 3y \\ x &= y. \end{aligned}$$

Thus, f is injective. □

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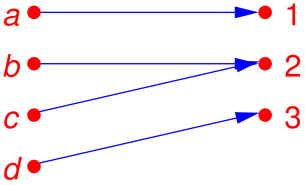
Surjective Functions

Definition

A function $f : A \to B$ is **surjective** (or **onto**) if for every $b \in B$ there is an $a \in A$ with $f(a) = b$. Such a function is called an **surjection**.

Suppose $A = \{a, b, c, d\}$.

$B = \{1, 2, 3\}$



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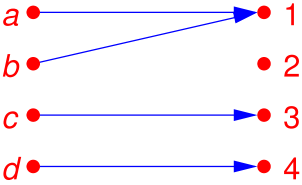

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Surjective Functions

To show a function $f : A \to B$ is surjective we need to establish that for each $b \in B$ there is an $a \in A$ such that $f(a) = b$.

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Proof.

Suppose $b \in \mathbb{Z}$. Then $f(a) = b$ if $a + 8 = b$ for some $a \in \mathbb{Z}$. However, $a = b - 8$ and since $b - 8$ is an integer, we know that the necessary a does exist. Therefore f is surjective. □

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Show that the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x + 8$ is surjective.

Proof.

Suppose $b \in \mathbb{Z}$. Then $f(a) = b$ if $a + 8 = b$ for some $a \in \mathbb{Z}$. However, $a = b - 8$ and since $b - 8$ is an integer, we know that the necessary a does exist. Therefore f is surjective. □

Show that the function $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = x^2$ is not surjective.

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Show that the function $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = x^2$ is not surjective.

Proof.

We merely need to find a counterexample. Suppose $b = 3$. There is no integer a such that $a^2 = 3$ since 3 is not a perfect square. Thus, g(x) is not surjective. □

Bijective Functions

As we've seen, it is possible for a function to be both injective and surjective.

Definition

A function f is **bijective** if it is both injective and surjective. Such a function is called a **bijection** (or a **one-to-one correspondence**).

Construct a function on the integers (from $\mathbb{Z}$ to $\mathbb{Z}$) that is

1. bijective.
2. surjective but not injective.
3. injective but not surjective.
4. neither injective nor surjective.

The Pigeonhole Principle

Theorem (Pigeonhole Principle)

If $k+1$ *or more objects are placed into* k *boxes, then at least one box contains more than one object.*

Example

In a group of 13 or more people, at least two of them have a birthday in the same month.

Example

Any set containing two digit numbers can have at most 90 elements (assuming the first digit is nonzero). If it had any more then an element would have to be repeated, violating the definition of a set.

The Pigeonhole Principle

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Hint: The key to this is to figure out the maximum number of people possible such that no month has more than nine people born in it.

The Pigeonhole Principle

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Hint: The key to this is to figure out the maximum number of people possible such that no month has more than nine people born in it.

Answer: The largest number of people that could be distributed so that no more than nine of them were born in the same month would be $9 \cdot 12 = 108$.

If we add 1 to this number then we know that at least one month will have the birthdays of at least 10 people. So the answer is 109 people is the minimum number of people necessary to guarantee that at least 10 of the people were born in the same month.

The Pigeonhole Principle

The Pigeonhole Principle for Sets

Suppose A and B are finite sets and $f : A \to B$ is any function. Then

1. If $|A| > |B|$, then f is not injective.
2. If $|A| < |B|$, then f is not surjective.

Proposition

Give a set of six integers, at least two of them will have the same remainder when divided by 5.

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Proposition

Give a set of six integers, at least two of them will have the same remainder when divided by 5.

Proof.

Let the set of six integers be called A. We know $\mathbb{Z}_5$ is the set of equivalence classes modulo 5, where each equivalence class contains all numbers that have the same remainder when divided by 5. Since $|\mathbb{Z}_5| = 5$, we have $|A| > |\mathbb{Z}_5|$. By the Pigeonhole principle we see that any function from A to $\mathbb{Z}_5$ cannot be injective, so at least two numbers in A must have the same remainder when divided by 5. □

The Pigeonhole Principle

Proposition

Given a set of any six natural numbers, the sum or difference of two of them is divisible by 9.

Proof.

Suppose A is a set of any six natural numbers. There are two possibilities:

First, suppose $\exists a, b \in A$ for which $a \equiv b \pmod 9$. Then $9|(a-b)$ and we are done!

Now suppose $\forall a, b \in A$ we find $a \not\equiv b \pmod 9$. Since no two of the numbers in A have the same remainder when divided by 9, the remainders form a set of six distinct numbers from $R = \{0, 1, 2, \ldots, 8\}$. There are four subsets of R consisting of pair of numbers that sum to 9; $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, and $\{4, 5\}$. Our proof will be complete if we can show that any set containing six numbers from R is guaranteed to contain one of these four subsets.

The Pigeonhole Principle

Proof.

(Continued)

Suppose we wanted to try and pick six numbers from R so that none of the subsets of S can be formed. We first pick four numbers: either 1 or 8, 2 or 7, 3 or 6, and 4 or 5. Our only choice for the fifth number is 0. As soon as we try and pick a sixth number, however, we unavoidably must pick a number that completes a pair.

Let $\{m, n\}$ be this pair. Then we know that $a, b \in A$ exist such that $a = 9j + m$ and $b = 9k + n$ for some integers j, k. Then

$$a + b = 9(j + k) + (m + n) = 9(j + k) + 9 = 9(j + k + 1)$$

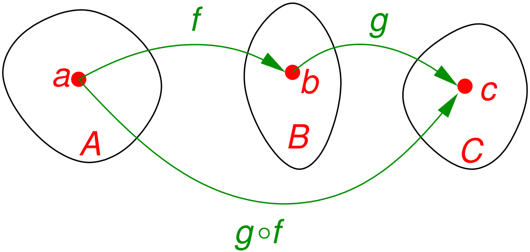
so $9 | (a + b)$.

Thus, any set of six natural numbers contains a pair of number such that the sum or difference of them is divisible by 9. □

Composition

Definition

Suppose $f : A \to B$ and $g : B \to C$ are functions with the property that the codomain of f equals the domain of g. The **composition** of f with g is another function, denoted $g \circ f$ and defined as $g \circ f(x) = g(f(x))$ for all $x \in A$. This function sends elements of A to elements of C, so $g \circ f : A \to C$.



Composition Example

Suppose

$$A = \{a, b, c\} \qquad B = \{1, 2\} \qquad C = \{\alpha, \beta, \gamma\}$$

with functions $f : A \to B$ and $g : B \to C$ defined as

$$f = \{(a, 1), (b, 2), (c, 1)\}$$
$$g = \{(1, \beta), (2, \gamma)\}.$$

The function $g \circ f : A \to C$ is

$$g \circ f = \{(a, \beta), (b, \gamma), (c, \beta)\}$$

Note:

- α is not in the range of g so α is not in the range of $g \circ f$.
- The composition of g with f, $f \circ g$, is not defined, since the codomain of g is not a subset of the domain of f.

Composition not Commutative

This last point is worth another look. In general, even when it is defined, *composition is not commutative*.

Suppose $f(x) = 2x$ and $g(x) = x + 5$ are functions defined on the integers so $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$. In this case both $g \circ f$ and $f \circ g$ are defined. However

$$g \circ f(x) = g(f(x)) = g(2x) = 2x + 5$$
$$f \circ g(x) = f(g(x)) = f(x + 5) = 2(x + 5).$$

Composition: Injection and Surjection

Theorem

Suppose $f : A \to B$ *and* $g : B \to C$*. If both* f *and* g *are injective, then* $g \circ f$ *is injective. If both* f *and* g *are surjective, then* $g \circ f$ *is surjective.*

The proof of this is given in our text.

Composition: Injection and Surjection

Proposition

Suppose $f : A \to B$ and $g : B \to C$. If both g and $g \circ f$ are injective, then f is injective.

Proof.

Suppose $f : A \to B$ and $g : B \to C$ and both g and $g \circ f$ are injective. Then $g(f(x)) = g(f(y))$ implies $x = y$ since $g \circ f$ is injective, and also implies $f(x) = f(y)$ since g is injective. But then we know that $x = y$ when $f(x) = f(y)$, so we can conclude that f is injective. □

Identity Function

Definition

Given a set A, the **identity function** on A is the function $i_A : A \to A$ defined as $i_A(x) = x$ for every $x \in A$.

Inverse Functions

Suppose we have a function $f : A \to B$. Under what conditions is there a function that "undoes" what f does?

That, is, if f maps $a \in A$ to $b \in B$, what must be true about f so that there is a function that maps b back to a?

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1. f must be *injective* so that only one a maps to a given b.
2. f must be *surjective* so that every b can be mapped back to some a.

So, to be able to always "undo" the mapping $f : A \to B$, f must be *bijective*.

Definition

Suppose $f : A \to B$ is bijective. The **inverse function of** f is the function that assigns to each element $b \in B$ an element $a \in A$ such that $f(a) = b$. We denote the inverse function $f^{-1} : B \to A$ and can write

$$f^{-1}(b) = a \quad \text{when} \quad f(a) = b.$$

The Identity Function

Definition

Given a set A, the **identity function** on A is the function $i_A : A \to A$ defined as $i_A(x) = x$ for every $x \in A$.

The identify function is bijective, and so is invertible. In fact, it is its own inverse.

The identify function plays an important role, much as the multiplicative identity 1 and additive identity 0 play important roles when working with the real numbers. In particular, if $f : A \to B$ is invertible, then the inverse is $f^{-1} : B \to A$ and

$$i_A(x) = f^{-1}(f(x)) = (f^{-1} \circ f)(x), \text{ and}$$
$$i_A(x) = f(f^{-1}(x)) = (f \circ f^{-1})(x).$$

Finding the Inverse of a Function

Assuming the bijective function $f : A \to B$ is given as a formula operating on elements of A, we can in principle find a formula for f^{-1} using algebraic manipulation.

As a very simple example, consider $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x - 4$ for each $x \in \mathbb{Z}$. This is bijective, so we can find f^{-1}.

$$\begin{aligned} y &= f(x) = x - 4 \\ x &= y + 4 \end{aligned}$$

Thus $x = f^{-1}(y) = y + 4$. Note that it doesn't matter what symbol is used for y, and we are so used to thinking of x as the input value that we usually would rewrite this as $f^{-1}(x) = x + 4$.