Logic

MAT231

Transition to Higher Mathematics

Fall 2014

Outline

Statements

Definition

A **statement** or **proposition** is a sentence or mathematical expression that is either true or false.

Statements must be either true or false; they cannot be both.

We may not know the truth value of a statement and still know that it must be either true or false.

Example

Consider: "It is snowing at the south pole." This is a statement. We know it is either true or false, but we don't know which without doing some legwork.

Statements

Example

The following are all statements (propositions):

- $2 + 2 = 4$
- I am an American.
- My hair is blue.

None of the following are statements (propositions):

- $6 + 8$
- Simon says "sit down."
- Do you want to go to the store?

Naming statements

We can name statements to make them easier to work with. For example:

- P: The sum of an even integer and an odd integer is odd.
- Q: $\mathbb{Z} \subseteq \mathbb{Q}$
- R: There is at least one rational number between any two rational numbers

The value of naming statements will become apparent shortly.

Statements and open sentences

Sometimes we may need to have a “statement with a variable.”

- $P(x)$: If x is even, then x^2 is even.
- $Q(x)$: x is even.
- $R(x, y)$: $x < y$.

Here $P(x)$ is a statement with a variable x. It is always true, regardless of the value of x.

The truth of $Q(x)$, however, depends on the value of x. This is called a **propositional function** or an **open sentence**.

More than one variable may be present, as in $R(x, y)$. The truth of this open sentence can only be determined when both x and y are known.

Logical And (Conjunction)

Suppose we want to say that two statements are both true. For example, “the number 5 is odd” and “the number 7 is prime.” If

- P: the number 5 is odd.
- Q: the number 7 is a prime.

then the statement “P and Q” makes sense. Using only mathematical symbols, this is written

$$P \wedge Q.$$

This operation is called a **logical and** or a **conjunction**.

Logical Or (Disjunction)

Sometimes we need to say that one or another statement is true. Consider

- P: Adam's eyes are blue.
- Q: Eve's hair is brown.

The statement

$$P \vee Q$$

is read "P or Q" and means that either P is true, Q is true, or both are true. This statement is still true if Adam's eyes are blue *and* Eve's hair is brown. This operation is a **logical or** or a **disjunction**.

Notice that statement is different from "Either P or Q is true," which implies that P is true or Q is true, but not both.

Logical Not (Negation)

The statement "P: 2 is the only even prime number" is true.

The statement "2 is not the only even prime number" must therefore be false.

We indicate **negation** of P by $\sim P$ or $\neg P$.

$\sim P$ always has the opposite truth value than P has.

Truth Tables

A truth table is a useful tool with which to analyze statements. It works by listing all possible truth values of a set of statements.

The following are truth tables for And, Or, Not:

And (Conjunction)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
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Notice that the number of rows in the truth table depends on the number of statements (propositions). Tables with both P and Q have four rows while the table with just P only has two rows.

Conditional Statements

A **conditional statement** or **implication** is a statement that can be written in the form "if P, then Q."

- If $x \geq 3$, then $x^2 \geq 9$.
- s is even if $s = 2x$ for some integer x.
- $3 + t = 7$ implies $t = 4$.

The second statement has the form "Q if P" and is equivalent to "if $s = 2x, x \in \mathbb{Z}$, then s is even."

We use the notation $P \Rightarrow Q$ for conditional statements, and usually read them as "if P, then Q" or "P implies Q."

Truth Tables for Conditional Statements

What should the truth table for $P \Rightarrow Q$ look like?

- When both P and Q are true we should have $P \Rightarrow Q$ be true.
- Similarly, when P is true but Q is false we need $P \Rightarrow Q$ to be false, since it is not the case that P being true implies Q is true.

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- Similarly, when P is true but Q is false we need $P \Rightarrow Q$ to be false, since it is not the case that P being true implies Q is true.

In fact, the truth table for $P \Rightarrow Q$ is

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The last two lines may need a little explanation...

Truth Tables for Conditional Statements

First, let's agree on the following points

1. Statements are either true or false—they cannot be both.
2. If a statement is not false, then it must be true, and vice versa.

Consider the statement

$$\text{if } x \text{ is odd, then } 3x \text{ is odd.}$$

What should we do when x is even?

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$$\text{if } x \text{ is odd, then } 3x \text{ is odd.}$$

What should we do when x is even?

Can we get any information from this statement when x is even? In particular, is the statement *false* when x is even?

No, the statement is not false when x is even. Since it isn't false, it must be true!

And and Or are commutative

Using truth tables it is easy to verify that $P \wedge Q$ is equal to $Q \wedge P$ and $P \vee Q$ is equal to $Q \wedge P$.

Since the truth values shown in the third column of each pair of tables match, we know that the corresponding statements are equivalent.

And (Conjunction)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$Q \wedge P$
T	T	T
T	F	F
F	T	F
F	F	F

Or (Disjunction)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$Q \vee P$
T	T	T
T	F	T
F	T	T
F	F	F

Converse

Conditional statements are not commutative:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$Q \Rightarrow P$
T	T	T
T	F	T
F	T	F
F	F	T

The statement "$Q \Rightarrow P$" is called the **converse** of $P \Rightarrow Q$.

Example

"If it is raining, then we will get wet" is a conditional statement. Its converse is "If we get wet, then it is raining." These are not equivalent statements: we could be wet for any number of reasons (water balloons, falling in a pool, etc.) when it's not raining.

Contrapositive

For completeness, we examine one more conditional statement, the *contrapositive*.

The **contrapositive** of "$P \Rightarrow Q$" is "$\sim Q \Rightarrow \sim P$" and has the truth table shown on the right.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Since the rightmost columns in these truth tables match, we see that the contrapositive is equivalent to the original implication.

This will be important to us when we are trying to write proofs.

Biconditional Statements

Although in general a conditional statement and its converse are not equivalent, this can be true for particular conditional statements.

A **biconditional statement** formed by

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

This is a strong statement; if either P or Q are true, then we know the other is also true. If either is false, the other is as well.

We denote biconditional statements with $P \Leftrightarrow Q$ and would read this as "P if and only if Q."

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional Statement Truth Table

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Compound Statements

A **compound statement** is one involving more than one simple statement. For example $P \wedge Q$ is a compound statement made up of the two simple statements P and Q.

A compound statement is a **tautology** if it is always true and it is a **contradiction** if it is always false; otherwise it is called a **contingency**.

The statements P and Q are **logically equivalent** if $P \Leftrightarrow Q$ is a tautology. We use the notation $P \equiv Q$ to denote logical equivalence (our text uses $P = Q$).

Compound Statements

Consider the statement

$$(P \vee Q) \wedge \sim(P \wedge Q)$$

How would you express this statement in English?

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How would you express this statement in English?

P or Q is true, and it is not the case that both P and Q are true.

The truth table for this operation is

P	Q	$P \vee Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Some of you may recognize this as the *exclusive or* operation, sometimes denoted $P \oplus Q$.

Logical Equivalence using Truth Tables

Recall that P is logically equivalent to Q if $P \Leftrightarrow Q$ is a tautology (always true).

The biconditional statement truth table is

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Clearly, $P \Leftrightarrow Q$ is true whenever P and Q have the same truth values and it is false when the truth values differ.

Thus, two statements are logically equivalent if their corresponding columns in *the same truth table* have identical entries.

Logical Equivalence using Truth Tables

Example

Use a truth table to verify $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$.

Solution:

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
T	T	T	**F**	F	F	**F**
T	F	F	**T**	F	T	**T**
F	T	F	**T**	T	F	**T**
F	F	F	**T**	T	T	**T**

The columns for $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ match, so these statements are logically equivalent.

Exercise

Use a truth table to prove the following logical equivalence.

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

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Solution:

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**T**	T	F	**T**
T	F	T	T	**T**	F	T	**T**
T	F	F	F	**F**	F	F	**F**
F	T	T	T	**F**	F	F	**F**
F	T	F	T	**F**	F	F	**F**
F	F	T	T	**F**	F	F	**F**
F	F	F	F	**F**	F	F	**F**

Thus, "logical and" distributes over "logical or."

Exercise

Use a truth table to prove the following logical equivalence.

$$P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

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Use a truth table to prove the following logical equivalence.

$$P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

Solution:

P	Q	$P \Leftrightarrow Q$	$P \wedge Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$
T	T	**T**	T	F	F	F	**T**
T	F	**F**	F	F	T	F	**F**
F	T	**F**	F	T	F	F	**F**
F	F	**T**	F	T	T	T	**T**

Logical Equivalences

$$P \wedge T \equiv P \qquad \text{Identity Laws}$$
$$P \vee F \equiv P$$

$$P \vee T \equiv T \qquad \text{Domination Laws}$$
$$P \wedge F \equiv F$$

$$P \wedge P \equiv P \qquad \text{Idempotent Laws}$$
$$P \vee P \equiv P$$

$$\sim(\sim P) \equiv P \qquad \text{Double Negation Law}$$

Logical Equivalences

$$P \wedge Q \equiv Q \wedge P \qquad \text{Commutative Laws}$$
$$P \vee Q \equiv Q \vee P$$

$$(P \vee Q) \vee R \equiv P \vee (Q \vee R) \qquad \text{Associative Laws}$$
$$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$$

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) \qquad \text{Distributive Laws}$$
$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

$$\sim(P \wedge Q) \equiv \sim P \vee \sim Q \qquad \text{DeMorgan's Laws}$$
$$\sim(P \vee Q) \equiv \sim P \wedge \sim Q$$

Open Sentences

Recall that an open sentence is represented by expressions like $P(x)$ or $Q(x, y)$, where x and y represent variables.

The truth of these statements may or may not depend on the variable.

When is an open statement (propositional function) become a statement (proposition)?

1. once all variables are assigned values,
2. when the truth value can be determined for all values in the **universe of discourse** (essentially the domain of the the variables).

Quantifiers

Sometimes it will be important to ask questions like

- "For all values of x, is $P(x)$ true?"
- "for some value of x, is $P(x)$ true?"

To answer these questions, we use **quantifications**.

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A **universal quantification** is written

$$\forall x \in U, P(x)$$

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An **existential quantification** is written

$$\exists x \in U, P(x)$$

and is true if $P(x)$ is true for at least one $x \in U$.

Example

Let $P(x) : x^2 > 0$ and the universe of discourse be $\mathbb{R}$. Determine the truth values of both the universal and existential quantifications.

Solution:

- $\forall x \in \mathbb{R}, x^2 > 0$ is false because $x^2 = 0$ when $x = 0$.
- $\exists x \in \mathbb{R}, x^2 > 0$ is true because $x^2 = 1$ when $x = 1$. (Many other examples could be used)

An Important Convention

Consider the statement about an integer x:

> If x is a multiple of 2 and a multiple of 5, then x is a multiple of 10.

It is a common convention in mathematics that this is understood to mean

> **For all integers** x, if x is a multiple of 2 and a multiple of 5, then x is a multiple of 10.

In symbols, when you see a statement about sets of the form

$$P(x) \Rightarrow Q(x),$$

you should understand this to mean

$$\forall x, P(x) \Rightarrow Q(x).$$

A simple example

Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the universe of discourse for x is the set of students in a class. Express each of the following quantifications in English:

- $\exists x, P(x)$
- $\forall x, P(x)$
- $\exists x, \sim P(x)$
- $\forall x, \sim P(x)$

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- $\exists x, P(x)$: "There is a student who spends more than five hours each weekday in class."
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- $\exists x, P(x)$: "There is a student who spends more than five hours each weekday in class."
- $\forall x, P(x)$: "Every student spends more than five hours each weekday in class."
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- $\exists x, \sim P(x)$: "There is a student that spends five or fewer hours each weekday in class."
- $\forall x, \sim P(x)$

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- $\exists x, \sim P(x)$: "There is a student that spends five or fewer hours each weekday in class."
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Multiple Quantifiers

When there are two or more quantifiers present, the order of the quantifiers is significant.

Let $T(x, y)$ be the open sentence "x is at least as tall as y" where the universe of discourse consists of three students

Garth	5' 9"
Erin	5' 6"
Marty	6' 0"

Express in words the following propositions and determine their truth value:

- $\forall x, \forall y, T(x, y)$
- $\forall x, \exists y, T(x, y)$
- $\exists x, \forall y, T(x, y)$
- $\exists x, \exists y, T(x, y)$

Multiple Quantifiers

$T(x, y)$: "x is at least as tall as y."

Garth 5' 9", Erin 5' 6", Marty 6' 0"

- $\forall x, \forall y, T(x, y)$
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$T(x, y)$: "x is at least as tall as y."

Garth 5' 9", Erin 5' 6", Marty 6' 0"

- $\forall x, \forall y, T(x, y)$: "Every student is at least as tall as every student."
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Multiple Quantifiers

$T(x, y)$: "x is at least as tall as y."

Garth 5' 9", Erin 5' 6", Marty 6' 0"

- $\forall x, \forall y, T(x, y)$: "Every student is at least as tall as every student." This, of course, is false.
- $\forall x, \exists y, T(x, y)$
- $\exists x, \forall y, T(x, y)$
- $\exists x, \exists y, T(x, y)$

Multiple Quantifiers

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- $\forall x, \forall y, T(x, y)$: "Every student is at least as tall as every student." This, of course, is false.
- $\forall x, \exists y, T(x, y)$: "All students are at least as tall as some student."
- $\exists x, \forall y, T(x, y)$
- $\exists x, \exists y, T(x, y)$

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- $\forall x, \exists y, T(x,y)$: "All students are at least as tall as some student." This is true.
- $\exists x, \forall y, T(x,y)$: "There is a student that is at least as tall as every student."
- $\exists x, \exists y, T(x,y)$

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- $\forall x, \exists y, T(x, y)$: "All students are at least as tall as some student." This is true.
- $\exists x, \forall y, T(x, y)$: "There is a student that is at least as tall as every student." This is true.
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- $\forall x, \exists y, T(x, y)$: "All students are at least as tall as some student." This is true.
- $\exists x, \forall y, T(x, y)$: "There is a student that is at least as tall as every student." This is true.
- $\exists x, \exists y, T(x, y)$: "There is a student that is at least as tall as some student."

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- $\forall x, \exists y, T(x, y)$: "All students are at least as tall as some student." This is true.
- $\exists x, \forall y, T(x, y)$: "There is a student that is at least as tall as every student." This is true.
- $\exists x, \exists y, T(x, y)$: "There is a student that is at least as tall as some student." This is true as well.

Expressing Quantifiers

Let $L(x, y)$ be the sentence "x loves y" and let the universe of discourse be all people. Express the following statements as quantifications.

- Everybody loves Jerry.

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- Everybody loves Jerry. $\forall x, L(x, \text{Jerry})$
- Everybody loves somebody.

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Let $L(x, y)$ be the sentence "x loves y" and let the universe of discourse be all people. Express the following statements as quantifications.

- Everybody loves Jerry. $\forall x, L(x, \text{Jerry})$
- Everybody loves somebody. $\forall x, \exists y, L(x, y)$

Expressing Quantifiers

Let $L(x, y)$ be the sentence "x loves y" and let the universe of discourse be all people. Express the following statements as quantifications.

- Everybody loves Jerry. $\forall x, L(x, \text{Jerry})$
- Everybody loves somebody. $\forall x, \exists y, L(x, y)$
- There is somebody that everybody loves.

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- Everybody loves somebody. $\forall x, \exists y, L(x, y)$
- There is somebody that everybody loves. $\exists y, \forall x, L(x, y)$
- There is somebody that Lydia does not love.

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- There is somebody that Lydia does not love. $\exists y, \sim L(\text{Lydia}, y)$

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- There is somebody that no one loves. $\exists y, \forall x, \sim L(x, y)$

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- There is somebody that Lydia does not love. $\exists y, \sim L(\text{Lydia}, y)$
- There is somebody that no one loves. $\exists y, \forall x, \sim L(x, y)$
- Everyone loves himself or herself.

Expressing Quantifiers

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- There is somebody that no one loves. $\exists y, \forall x, \sim L(x, y)$
- Everyone loves himself or herself. $\forall x, L(x, x)$

Expressing Quantifiers

Here are some more...

Let $L(x, y)$ be the sentence "x loves y" and let the universe of discourse be all people. Express the following statements as quantifications.

- There is exactly one person that everybody loves.

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  $\exists y, [(\forall x, L(x, y)) \wedge (\forall z, (\forall x, L(x, z) \Rightarrow y = z))]$

Expressing Quantifiers

Here are some more...

Let $L(x, y)$ be the sentence "x loves y" and let the universe of discourse be all people. Express the following statements as quantifications.

- There is exactly one person that everybody loves.
  $\exists y, [(\forall x, L(x, y)) \wedge (\forall z, (\forall x, L(x, z) \Rightarrow y = z))]$
- There are two people that Lynn loves.

Expressing Quantifiers

Here are some more...

Let $L(x, y)$ be the sentence "x loves y" and let the universe of discourse be all people. Express the following statements as quantifications.

- There is exactly one person that everybody loves.
  $\exists y, [(\forall x, L(x, y)) \wedge (\forall z, (\forall x, L(x, z) \Rightarrow y = z))]$
- There are two people that Lynn loves.
  $\exists x, \exists y, [\forall z, (L(\text{Lynn}, z) \Rightarrow (x = z \vee y = z)) \wedge x \neq y]$

Expressing Quantifiers

Here are some more...

Let $L(x, y)$ be the sentence "x loves y" and let the universe of discourse be all people. Express the following statements as quantifications.

- There is exactly one person that everybody loves.
  $\exists y, [(\forall x, L(x, y)) \wedge (\forall z, (\forall x, L(x, z) \Rightarrow y = z))]$
- There are two people that Lynn loves.
  $\exists x, \exists y, [\forall z, (L(\text{Lynn}, z) \Rightarrow (x = z \vee y = z)) \wedge x \neq y]$
- There is someone who loves no one besides himself.

Expressing Quantifiers

Here are some more...

Let $L(x, y)$ be the sentence "x loves y" and let the universe of discourse be all people. Express the following statements as quantifications.

- There is exactly one person that everybody loves.
  $\exists y, [(\forall x, L(x, y)) \wedge (\forall z, (\forall x, L(x, z) \Rightarrow y = z))]$
- There are two people that Lynn loves.
  $\exists x, \exists y, [\forall z, (L(\text{Lynn}, z) \Rightarrow (x = z \vee y = z)) \wedge x \neq y]$
- There is someone who loves no one besides himself.
  $\exists x, \forall y, [L(x, y) \Rightarrow x = y]$

Consider the following theorem where $\mathbb{C}$ is the set of complex numbers:

Theorem (Fundamental Theorem of Algebra)

If $f(x)$ is a polynomial function of degree at least 1, then there exists a number $c \in \mathbb{C}$ such that $f(c) = 0$.

How could this be written in mathematical notation?

Consider the following theorem where $\mathbb{C}$ is the set of complex numbers:

Theorem (Fundamental Theorem of Algebra)

If $f(x)$ is a polynomial function of degree at least 1, then there exists a number $c \in \mathbb{C}$ such that $f(c) = 0$.

How could this be written in mathematical notation?

Let $\mathbb{P}$ be the set of polynomial functions of a single variable and $\deg(f)$ be the degree of polynomial f. Then we can say

$$(f \in \mathbb{P}) \wedge (\deg(f) \geq 1) \Rightarrow \exists c \in \mathbb{C}, f(c) = 0.$$

Notice how we defined additional notation that allowed us to restate the theorem without confusion.

An important point...

Theorem

Every integer multiple of 6 is an integer multiple of 2 and a integer multiple of 3.

If we let S_6 be the set of integer multiples of 6 and S_2 and S_3 be the sets of integer multiples of 2 and 3 respectively, the theorem can be stated

$$(x \in S_6) \Rightarrow (x \in S_2 \wedge x \in S_3) \qquad \text{or} \qquad \forall x \in S_6, (x \in S_2 \wedge x \in S_3)$$

In fact, if S is a set and $Q(x)$ is a statement about x for each $x \in S$, the following two forms are equivalent and my be used interchangeably:

$$(x \in S) \Rightarrow Q(x) \qquad \text{or} \qquad \forall x \in S, Q(x)$$

DeMorgan's Laws

Recall we can use DeMorgan's Laws to help us negate conjunctions and disjunctions:

$$\sim(P \wedge Q) \equiv \sim P \vee \sim Q \qquad \text{and} \qquad \sim(P \vee Q) \equiv \sim P \wedge \sim Q$$

Example

Negate the statement "Let x and y both be even integers."

First we note that this can be written as "Let x be an even integer and y be an even integer." Now we can apply DeMorgan's Law:

$$\begin{aligned} \sim((x \text{ is even}) \wedge (y \text{ is even})) &\equiv \sim(x \text{ is even}) \vee \sim(y \text{ is even}) \\ &\equiv (x \text{ is odd}) \vee (y \text{ is odd}). \end{aligned}$$

So, the negation of our original statement would be "Let x be odd or y be odd."

Negating Quantifications

Consider the universal quantification $\forall x \in S, P(x)$ for some set S and open sentence $P(x)$. The negation of this statement is

$$\sim(\forall x \in S, P(x))$$

Writing this in words, we have something like

> "It is not the case that $P(x)$ is true for all values x from the set S."

This means that there is at least one $x \in S$ for which $P(x)$ is not true. We can write this as $\exists x \in S, \sim P(x)$.

Fact

The negation of universal and existential quantifications is given by

$$\sim(\forall x \in S, P(x)) \equiv \exists x \in S, \sim P(x)$$
$$\sim(\exists x \in S, P(x)) \equiv \forall x \in S, \sim P(x)$$

Expressing Quantifiers

Let's return to our open sentence

$$L(x, y): \ x \text{ loves } y$$

and let the universe of discourse be all people. Express the following statements as quantifications.

Before we noted that "There is somebody that Lydia does not love" could be written as $\exists y, \sim L(\text{Lydia}, y)$. We see now that this could also be written as $\sim(\forall y, L(\text{Lydia}, y))$.

How could we write the statement "nobody loves everybody" as a quantification? In this case it may be easier to form the opposite statement and then negate it. The opposite statement could be written "somebody loves everybody," which would be $\exists x, \forall y, L(x, y)$. Negating we have

$$\sim(\exists x, \forall y, L(x, y)) \qquad \text{or} \qquad \forall x, \exists y, \sim L(x, y).$$

Making Arguments

As we have seen, logical statements can be combined to produce more complicated compound statements.

Another way to “combine” them is while making an argument, through the process of **logical deduction**.

At it simplest, we want to be able to begin at some point, and through a sequence of **logical inferences**, move from one logical statement to another and ending up with the desired result.

The table on the next slides shows the most important **rules of inference**. We will use each of these at various points as we study proof techniques.

Rules of Inference

Rule of Inference	Name
$P \Rightarrow (P \vee Q)$	Addition
$(P \wedge Q) \Rightarrow P$	Simplification
$((P \Rightarrow Q) \wedge P) \Rightarrow Q$	Modus ponens
$((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$	Modus tollens
$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$	Hypothetical syllogism
$(P \vee Q) \wedge \sim P) \Rightarrow Q$	Disjunctive syllogism

Rules of Inference examples

Addition: It is sunny. Therefore it is either sunny or it is raining.

Simplification: It is sunny and it is hot. Therefore it is sunny.

Modus ponens: If it is sunny then it is hot. It is sunny. Therefore it is hot.

Modus tollens: If it is sunny then it is hot. It is not hot. Therefore it is not sunny.

Hypothetical syllogism: If it is sunny then it is hot. If it is hot then we want to go to the beach. Therefore if it is sunny then we want to go to the beach.

Disjunctive Syllogism: It is sunny or it is raining. It is not sunny. Therefore it is raining.