Relations

MAT231

Transition to Higher Mathematics

Fall 2014

Outline

Relation on a Set

Definition

A **relation** on a set A is a subset $R \subseteq A \times A$. We often abbreviate the statement $(x, y) \in R$ as $x \, R \, y$. The statement $(x, y) \notin R$ is abbreviated as $x \not R \, y$.

Suppose $A = \{1, 2, 3, 4\}$. The following are all relations on A:

- $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
- $S = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$
- $T = \{(3, 4)\}$
- $U = \{(1, 4), (2, 3), (2, 1)\}$

since each one is a subset of $A \times A$.

Relations may or may not have meaning associated with them. Two numbers are related by R if they are equal. We see that $x \, S \, y$ if x and y have the same parity.

The relationships expressed by T and U are not readily apparent.

Examples of Relations

Consider the relation "x is less than y" on the set $A = \{1, 2, 3, 4, 5, 6\}$. This could be expressed as $L = \{(x, y) : x, y \in A \text{ and } x < y\}$ or listed as

$$L = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),\\ (3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}.$$

Note that usually we use the symbol $<$ as the name for L.

Suppose there is another relation, D, defined as "x is related to y if $x|y$." This would be $\{(x, y) : x, y \in A \text{ and } x|y\}$ or

$$D = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,4),(2,6),\\ (3,3),(3,6),(4,4),(5,5),(6,6)\}$$

The symbol $|$ is usually used to denote this relation (as we have, somewhat recursively, in our definition of D).

Examples of Relations

Using the sets from the previous slide

$$\begin{aligned} L = \{&(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),\\ &(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\},\\ D = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,4),(2,6),\\ &(3,3),(3,6),(4,4),(5,5),(6,6)\}, \end{aligned}$$

consider

$$L \cap D = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,4),(2,6),(3,6)\}.$$

Here $(x, y) \in L \cap D$ if $x < y$ *and* $x|y$.

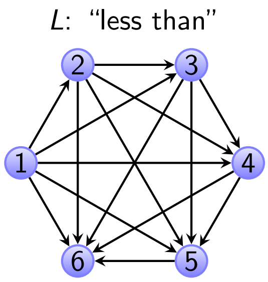
In contrast, the relation $L \cup D$ represents the relation where, if $(x, y) \in L \cup D$ then $x < y$ *or* $x|y$. This relation is

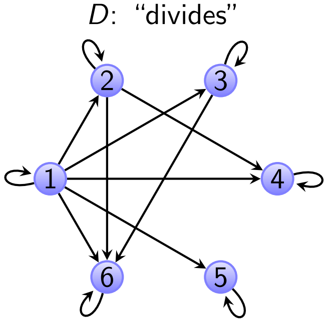
$$\begin{aligned} L \cup D = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),\\ &(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6),(4,4),\\ &(4,5),(4,6),(5,5),(5,6),(6,6)\}. \end{aligned}$$

Representing Relations

There are multiple ways relations can be represented:

- We have already seen that relations can be *enumerated*, i.e., since they are sets they can be listed.
- A *directed graph* can represent a relation R on a set A. Each *node* in the graph represents an element of A and an arrow from node x to node y indicates that $(x, y) \in R$. For example, using set A and relations L and D from the previous slides, we have





Properties of Relations

Definition

Suppose R is a relation on a set A.

1. Relation R is **reflexive** if $(x, x) \in R$ for every $x \in A$.
2. Relation R is **symmetric** if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in A$.
3. Relation R is **antisymmetric** if $(x, y) \in R$ and $(y, x) \in R$ implies that $x = y$ for all $x, y \in A$.
4. Relation R is **transitive** if whenever $(x, y) \in R$ and $(y, z) \in R$, then also $(x, z) \in R$.

Once again, suppose $A = \{1, 2, 3, 4, 5, 6\}$. Which of the above properties does the $<$ relation have? Does the $\leq$ relation have any different properties than $<$ has?

Which of these properties does the $\mid$ relation have?

Proving Properties of Relations

Proposition

The relation $|$ on the set $\mathbb{Z}$ is reflexive.

Proof.

Suppose $x \in \mathbb{Z}$. Since $x = 1 \cdot x$ we know that $x|x$. Thus, the relation $|$ is reflexive on $\mathbb{Z}$. □

Proving Properties of Relations

Proposition

The relation $|$ on the set $\mathbb{Z}$ is transitive.

Proof.

Suppose $x, y, z \in \mathbb{Z}$ and $x|y$ and $y|z$. Integers a and b must exist such that $y = ax$ and $z = by$. But then $z = b(ax) = (ab)x$ and so $x|z$. Thus, $|$ is transitive on the integers. □

Properties of Relations in Graphs

How do each of the properties of relations show up in graphs of relations?

- The graph of a **reflexive** relation will have a *loop edge* at each node.

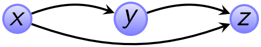
- The graph of a **symmetric** relation will not have an edge from x to y unless there is also an edge from y to x.

- The graph of a **antisymmetric** relation will not have any symmetric pairings. If an edge goes from x to y, there cannot be an edge from y to x.

Note: A graph if symmetric if there are no antisymmetric pairs. Similarly, a graph is antisymmetric if there are no symmetric pairs. *It is possible for a graph to be both symmetric and antisymmetric.*

Properties of Relations in Graphs

- The graph of a **transitive** relation will have an edge from x to z whenever there is an edge from x to y and an edge from y to z.



Examples

Consider the following relations on the set $\{1, 2, 3, 4\}$. Determine which ones are reflexive, symmetric, antisymmetric or transitive.

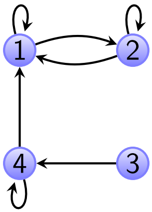
- $R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$
- $R_2 = \{(1,1),(1,2),(2,1)\}$
- $R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$
- $R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$
- $R_5 =$ $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$
- $R_6 = \{(3,4)\}$

R_1

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

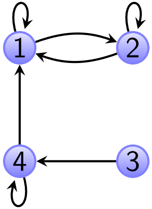
R_1

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$



R_1

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$



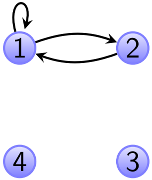
- Not reflexive since $(3,3)$ is not in R_1 (no loop edge on 3).
- Not symmetric since $(3,4) \in R_1$ but $(4,3) \notin R_1$.
- Not antisymmetric since both $(1,2)$ and $(2,1)$ are in R_1.
- Not transitive because $(3,4)$ and $(4,1)$ are both in R_1 but $(3,1)$ (a "short-cut" edge) is not.

R_2

$R_2 = \{(1,1),(1,2),(2,1)\}$

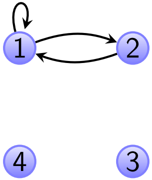
R_2

$R_2 = \{(1,1),(1,2),(2,1)\}$



R_2

$R_2 = \{(1,1),(1,2),(2,1)\}$



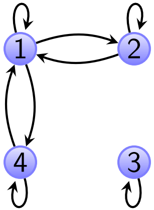
- Not reflexive; $(2,2) \notin R_2$.
- Symmetric; there are no non-symmetric connections.
- Not antisymmetric.
- Not transitive because $(2,1)$ and $(1,2)$ are both in R_2 but $(2,2)$ is not.

R_3

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

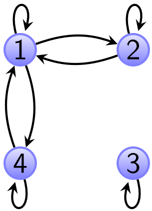
R_3

$$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$$



R_3

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$



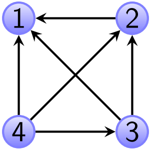
- Reflexive.
- Symmetric.
- Not antisymmetric.
- Not transitive; $(4,1),(1,2) \in R_3$ but $(4,2) \notin R_3$.

R_4

$$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$

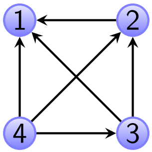
R_4

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$



R_4

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$



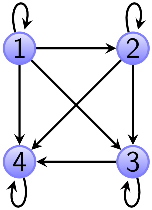
- Not reflexive.
- Not symmetric.
- Antisymmetric; no symmetric pairs.
- Transitive.

R_5

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

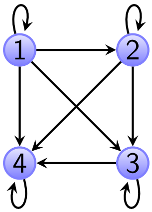
R_5

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$



R_5

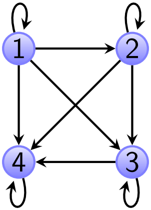
$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$



- Reflexive.
- Not symmetric.
- Antisymmetric; no symmetric pairs.
- Transitive.

R_5

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$



- Reflexive.
- Not symmetric.
- Antisymmetric; no symmetric pairs.
- Transitive.

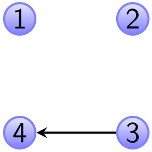
This is the “less than or equal to” relation on $\{1,2,3,4\}$.

R_6

$R_6 = \{(3, 4)\}$

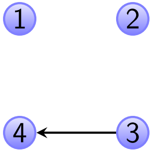
R_6

$R_6 = \{(3,4)\}$



R_6

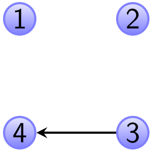
$R_6 = \{(3,4)\}$



- Not reflexive.
- Not symmetric.
- Antisymmetric.
- Transitive.

R_6

$R_6 = \{(3, 4)\}$



- Not reflexive.
- Not symmetric.
- Antisymmetric.
- Transitive.

This is transitive since there are now "double hops" in the graph without a short cut edge

Equivalence Relations

Consider the following relations on the set of people in this room

- $\{(a, b) : a$ and b were born in the same month$\}$,
- $\{(a, b) : a$ and b are the same sex$\}$,
- $\{(a, b) : a$ and b are from the the same state$\}$.

Observe that these relations are all reflexive, symmetric and transitive. Because of this they are all *equivalent* in some way.

Equivalence Relations

Definition

A relation R on a set A is an **equivalence relation** if it is reflexive, symmetric and transitive.

Suppose that R is a relation on $\mathbb{N}$ such that $(a, b) \in R$ if and only if $a < 5$ and $b < 5$. Is R an equivalence relation?

- Since $a = a$ it follows that if $a < 5$ then $(a, a) \in R$ so we know that R is reflexive.
- Suppose $(a, b) \in R$ so both $a < 5$ and $b < 5$. In this case certainly $(b, a) \in R$ so that R is symmetric.
- Finally, if $(a, b) \in R$ and $(b, c) \in R$ then both a and c are less than 5 so $(a, c) \in R$, showing that R is transitive.

Thus R is an equivalence relation.

Equivalence Classes

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class containing** a. This is denoted $[a]_R$ or just $[a]$ if it is clear what R is.

Some examples:

- Suppose $R = \{(a, b) : a \text{ and } b \text{ were born in the same month}\}$ and is defined on the set of people in this room. Then

$$[a] = \{b : b \text{ was born in the same month as } a\}.$$

- Suppose $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$. We can list the equivalence class for each element of A as

$$[1] = \{1, 2\}, \qquad [2] = \{1, 2\}, \qquad [3] = \{3, 4\}, \qquad [4] = \{3, 4\}$$

Equivalence Classes and Partitions

A **partition** of a set S is a collection of disjoint, nonempty subsets of S that have S as their union.

If $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ then one partition of S is

$$\{\{1, 2\}, \{3\}, \{4, 5, 6\}, \{7, 8\}\}$$

Notice that every element of S is in exactly one of the subsets.

The equivalence classes of a relation on a set A form a partition of A:

- The union of all the equivalence classes of A is equal to A, i.e.,

$$\bigcup_{a \in A} [a] = A.$$

- $[a] \cap [b] = \varnothing$ when $[a] \neq [b]$.

Equivalence Classes and Partitions

Theorem

Let R be an equivalence relation on A. The following statements are equivalent.

1. $a \, R \, b$
2. $[a] = [b]$
3. $[a] \cap [b] \neq \varnothing$

We can prove this using a standard approach. First show $1 \Rightarrow 2$, then show $2 \Rightarrow 3$, and finally show $3 \Rightarrow 1$.

Equivalence Classes and Partitions

Proof.

$1 \Rightarrow 2$: $a \, R \, b \Rightarrow [a] = [b]$.

$$\begin{aligned}
[a] &= \{c \in A : (a, c) \in R\} & \\
&= \{c \in A : (c, a) \in R\} & R \text{ is symmetric} \\
&= \{c \in A : (c, b) \in R\} & a \, R \, b \text{ and } R \text{ is transitive} \\
&= \{c \in A : (b, c) \in R\} & R \text{ is symmetric} \\
&= [b]
\end{aligned}$$

(proof continued on next slide)

Equivalence Classes and Partitions

Proof.

(Continued)

$2 \Rightarrow 3$: $[a] = [b] \Rightarrow [a] \cap [b] \neq \varnothing$.
Let $a, b \in A$ such that $[a] = [b]$. Then, since $a \in [a]$ we know that $a \in [b]$. This means $a \in [a] \cap [b]$ so $[a] \cap [b] \neq \varnothing$.

$3 \Rightarrow 1$: $[a] \cap [b] \neq \varnothing \Rightarrow a\,R\,b$.
Let $c \in [a] \cap [b]$, which is possible since we know this set is nonempty. Therefore $c \in [a]$ and $c \in [b]$ so $c\,R\,a$ and $c\,R\,b$. Since R is symmetric we know $a\,R\,c$ and since R is transitive we can conclude that $a\,R\,b$.

□

The Integers Modulo n

Proposition

Let $n \in \mathbb{N}$. The relation $\equiv \pmod{n}$ on the set $\mathbb{Z}$ is reflexive, symmetric, and transitive, i.e., it is an equivalence relation.

Proof.

To see that $\equiv \pmod{n}$ is reflexive, suppose $a \in \mathbb{Z}$ and note that $n|(a-a)$ since $n|0$. Therefore $a \equiv a \pmod{n}$.

To show $\equiv \pmod{n}$ is symmetric, suppose $a \equiv b \pmod{n}$ for some $a, b \in \mathbb{Z}$. Then $n|(a-b)$ so $a - b = kn$ for some $k \in \mathbb{Z}$. However, this means that $b - a = (-k)n$ so $n|(b-a)$ and therefore $b \equiv a \pmod{n}$.

(Continued next slide)

The Integers Modulo n

Proof.

(Continued)

Finally, to show $\equiv \pmod{n}$ is transitive, suppose $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ for some $a, b, c \in \mathbb{Z}$. Then

$$n|(a-b) \qquad \text{and} \qquad n|(b-c)$$

so

$$a = b + jn \qquad \text{and} \qquad b = c + kn$$

for some $j, k \in \mathbb{Z}$. From this we see that

$$a = (c + kn) + jn = c + (j+k)n$$

which shows that $n|(a-c)$. Thus $a \equiv c \pmod{n}$. □

The Integers Modulo 5

The equivalence relation $\equiv \pmod{n}$ for a given $n \in \mathbb{N}$ is particularly important in mathematics. In particular, this relation can be used to partition the integers. Suppose, for example, that $n = 5$. Then the following five sets are disjoint and their union is $\mathbb{Z}$:

$$\begin{aligned}
[0] &= \{x \in \mathbb{Z} : n|(x-0)\} = \{\ldots, -10, -5, 0, 5, 10, 15, \ldots\} \\
[1] &= \{x \in \mathbb{Z} : n|(x-1)\} = \{\ldots, -9, -4, 1, 6, 11, 16, \ldots\} \\
[2] &= \{x \in \mathbb{Z} : n|(x-2)\} = \{\ldots, -8, -3, 2, 7, 12, 17, \ldots\} \\
[3] &= \{x \in \mathbb{Z} : n|(x-3)\} = \{\ldots, -7, -2, 3, 8, 13, 18, \ldots\} \\
[4] &= \{x \in \mathbb{Z} : n|(x-4)\} = \{\ldots, -6, -1, 4, 9, 14, 19, \ldots\}
\end{aligned}$$

We can define a new set

$$\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$$

which we call the **integers modulo 5**.

The Integers Modulo 5

To get familiar with $\mathbb{Z}_5$, let's try a few simple operations.

Consider $2 \in [2]$ and $4 \in [4]$:

- $2 + 4 = 6$ and $6 \in [1]$.
- $2 \cdot 4 = 8$ and $8 \in [3]$.

Next consider $7 \in [2]$ and one from $19 \in [4]$:

- $7 + 19 = 26$ and $26 \in [1]$.
- $7 \cdot 19 = 133$ and $133 \in [3]$.

On the one hand, the sum of numbers from [2] and [4] was a number from [1] while on the other hand the product of numbers from [2] and [4] was a number from [3].

The Integers Modulo 5

Let's try two pairs from another set of equivalence classes, say [2] and [3].

$$2 + 3 = 5 \in [0] \qquad \text{and} \qquad 2 \cdot 3 = 6 \in [1].$$

Working with different numbers from the same classes we find

$$-3 + 3 = 0 \in [0] \qquad \text{and} \qquad -3 \cdot 3 = -9 \in [1].$$

Once again, it seems that when we add a number from [2] to a number from [3] we obtain a number from [0]. Similarly, when we multiply a number from [2] by a number from [3] we find the product is from [1].

The Integers Modulo 5

These examples suggest that we can define addition and multiplication for $\mathbb{Z}_5$ as

$$[a] + [b] = [a + b]$$
$$[a] \cdot [b] = [a \cdot b]$$

Notice that $[a]$ and $[b]$ are sets, not numbers. Notice also that by this definition addition and multiplication are *closed* on $\mathbb{Z}_5$.

Since addition and multiplication on the integers is commutative, it seems reasonable to expect that $[a] + [b] = [b] + [a]$ and $[a] \cdot [b] = [b] \cdot [a]$. It is not difficult to prove this.

Other properties from $\mathbb{Z}$ hold as well. For example,
$[a] \cdot ([b] + [c]) = [a] \cdot [b] + [a] \cdot [c]$

The Integers Modulo 5

Just as when you first learned addition and multiplication, it is helpful to construct addition and multiplication tables for $\mathbb{Z}_5$.

+	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

·	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]
[2]	[0]	[2]	[4]	[1]	[3]
[3]	[0]	[3]	[1]	[4]	[2]
[4]	[0]	[4]	[3]	[2]	[1]

Notice the patterns in each of the tables. In particular, notice that while the order changes, every row and every column contain all five values [0], [1], [2], [3], and [4].

The Integers Modulo 5

Since $-1 \in [4]$, we can define subtraction for $\mathbb{Z}_5$ as

$$[a] - [b] = [a] + (-1) \cdot [b] = [a] + [4] \cdot [b]$$

For example

$$\begin{aligned} [23] - [7] &= [3] + [4] \cdot [2] \\ &= [3] + [4 \cdot 2] \\ &= [3] + [3] \\ &= [1] \end{aligned}$$

which is consistent with $23 - 7 = 16 \in [1]$.

The Integers Modulo n

Returning to the general case, we can make the following definition.

Definition

Let $n \in \mathbb{N}$. The equivalence classes of the equivalence relation $\equiv \pmod{n}$ are $[0], [1], [2], \ldots, [n-1]$. The **integers modulo n** is the set $\mathbb{Z}_n = \{[0],[1],[2],\ldots,[n-1]\}$.

Elements of $\mathbb{Z}_n$ can be added by the rule $[a]+[b]=[a+b]$ and multiplied by the rule $[a]\cdot[b]=[a\cdot b]$.

It is important to note that $[a]$ and $[b]$ are sets and not numbers. As long as

$$[a]=[a'] \qquad \text{and} \qquad [b]=[b']$$

we should find that $[a]+[b]$ will equal $[a']+[b']$ regardless of which particular a' and b' are used.

Relations Between Sets

The relations we've seen so far have been relations *on* a set. We can also have relations between sets.

Definition

A **relation** from a set A to a set B is a subset $R \subseteq A \times B$.

Partial Orders

Definition

A relation R on a set A is a **partial order** if it is reflexive, antisymmetric and transitive.

Suppose that R is a relation on $\mathbb{N}$ such that $(a, b) \in R$ if and only if $a \leq b$. Is R a partial order?

- Since $a \leq a$ we know $(a, a) \in R$ so R is reflexive.
- Suppose $(a, b) \in R$ so $a \leq b$. For $b \leq a$ it must be that $a = b$ so $(b, a) \in R$ only when $a = b$. Thus R is antisymmetric.
- Finally, if $(a, b) \in R$ and $(b, c) \in R$ then $a \leq b \leq c$ so $a \leq c$ implying that $(a, c) \in R$, showing that R is transitive.

Thus R is a partial order.

We often use the symbol $\preceq$ for a partial order.

Posets

Definition

A set S together with a partial order R is called a **partially ordered set** or **poset** and is denoted (S, R).

Suppose $(a, b) \in R$ if $a|b$ for all $a, b \in \mathbb{Z}^+$.

- We know R is reflexive since $a|a$.
- If $a|b$ then $b = ka$ for some $k \in \mathbb{Z}^+$. If $b|a$ then $a = jb$ for some $j \in \mathbb{Z}^+$ and so $a = (jk)a$ which implies that $jk = 1$. Since both j and k are integers this is only possible if $j = 1$ and $k = 1$, which implies $a = b$. Thus R is antisymmetric.
- Finally, if $a|b$ and $b|c$ then $b = ka$ and $c = jb$ for some $j, k \in \mathbb{Z}^+$. This means $c = (jk)a$ so $a|c$, making R transitive.

Thus $(\mathbb{Z}^+, R)$ is a poset.

Comparability and Total Orders

Definition

The elements a and b of a poset $(S, \preceq)$ are called **comparable** if either $a \preceq b$ or $b \preceq a$. When a and b are elements of S such that neither $a \preceq b$ or $b \preceq a$, a and b are called **incomparable**.

Consider $(\mathbb{Z}, |)$. The numbers 3 and 6 are comparable since $3|6$ is true. The numbers 3 and 5 are not comparable since neither $3|5$ nor $5|3$ is true.

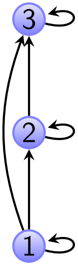
Definition

If $(S, \preceq)$ is a poset and every two elements of S are comparable, then $(S, \preceq)$ is called a **totally ordered set** and $\preceq$ is called a **total order**.

For example, $(\mathbb{Z}, \leq)$ is a totally ordered set while $(\mathbb{Z}, |)$ is not totally ordered.

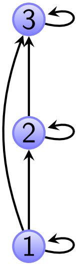
Hasse Diagrams

Consider the partial order on $S = \{1, 2, 3\}$ given by $(a, b) \in R$ if $a \leq b$. We could construct a directed graph of this relation as shown below.

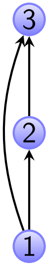


Hasse Diagrams

Original graph



Partial orders are reflexive so we can omit loop edges



Partial orders are transitive so we can omit "short cut" edges

If we always draw arrows up, we can omit arrowheads. This is called a **Hasse Diagram**.

Hasse Diagram Example

Exercise: Construct the Hasse Diagram for $(\{2, 5, 8, 10, 20\}, |)$.

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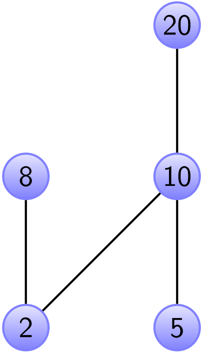
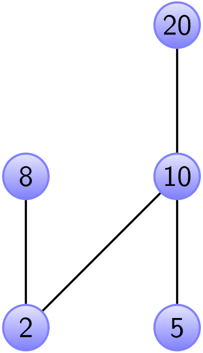


Hasse Diagram Example

Exercise: Construct the Hasse Diagram for $(\{2, 5, 8, 10, 20\}, |)$.

The **maximal elements** of this poset are the "tops;" In this case $\{8, 20\}$.



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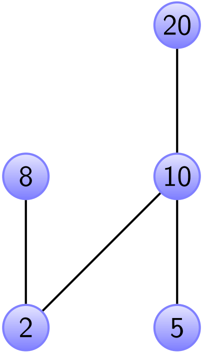

The **minimal elements** are the "bottoms;" in this case $\{2, 5\}$.

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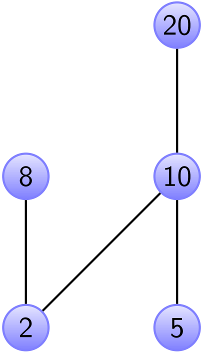
Exercise: Construct the Hasse Diagram for $(\{2, 5, 8, 10, 20\}, |)$.

The **maximal elements** of this poset are the "tops;" In this case $\{8, 20\}$.

If there is a single maximal element it is the **greatest element**. If there is more than one maximal element then there is no greatest element.



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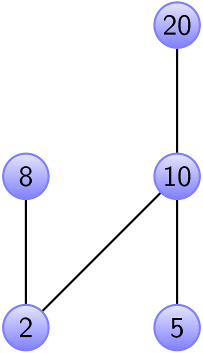


The **minimal elements** are the "bottoms;" in this case $\{2, 5\}$.

If there is a single minimal element it is the **least element**. If there is more than one minimal element then there is no least element.

Upper and Lower Bounds

Definition

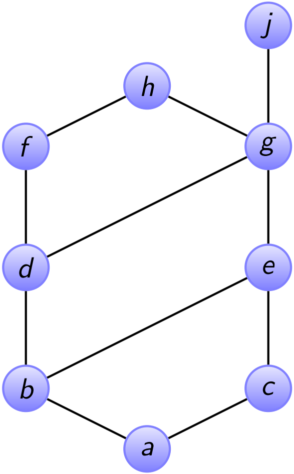
Let A be a subset of S from a poset (S, R). The **upper bounds** of A are the elements that are "above" every element in A in a Hasse diagram. This means that that every upper bound can be obtained by tracing up from each element in A.

The **lower bounds** of A are defined in a similar way; they are "below" all elements in A in a Hasse diagram.

The **least upper bound** of A is the "lowest" of the upper bounds of A and the **greatest lower bound** of A is the "highest" of the lower bounds of A.

Upper and Lower Bounds

Consider the poset with the following Hasse diagram.



If $A = \{a, b, c\}$ the upper bounds of A are $\{e, g, h, j\}$ and the least upper bound is e. The lower bounds of A are $\{a\}$ so the greatest lower bound is a.

If $B = \{d, e, g\}$ the lower bounds of B are $\{a, b\}$ and the greatest lower bound is b. The upper bounds are $\{g, h, j\}$ and the least upper bound is g.