Tridiagonal & Banded Systems

It is sometimes the case that a matrix of a linear system is sparse (many zeros) and sometimes the only nonzero entries are on the diagonal and several off-diagonals.

$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

Matrices with this form are called <u>banded</u> matrices. In this case

$$a_{ij} = 0 \quad \text{if} \quad |i-j| \geq k$$

If $k=2$ then there are only the diagonal, the subdiagonal and the superdiagonal. Matrices of this form are called <u>tridiagonal</u>.

Gaussian elimination for banded matrices is easy to perform if pivoting is not necessary (pivoting would change the banded structure.

A banded matrix also can require much less memory to store than a full matrix of the same dimensions.

Consider

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & c_3 & \\ & & a_3 & \ddots & \\ & & & \ddots & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

with a tridiagonal coefficient matrix

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Note that only $3n-2$ elements of the matrix need to be stored, rather than n^2 elements.

Performing elimination on the matrix begins with

$$d_2 \leftarrow d_2 - \frac{a_1}{d_1} c_1$$

In fact the entire process of reducing the matrix and RHS is done with:

for $i = 2$ to n

division done only once

$$d_i = d_i - \left(\frac{a_{i-1}}{d_{i-1}}\right) c_{i-1}$$

$$b_i = b_i - \left(\frac{a_{i-1}}{d_{i-1}}\right) b_{i-1}$$

endfor

Then $x_n = b_n / d_n$

and

for $i = n-1$ downto 1

$$x_i = (b_i - c_i \cdot x_{i+1}) / d_i$$

endfor

Total operation count is $5(n-1) + 1 + 3(n-1)$

$$= 8n - 7$$

$$= O(n)$$

This in in contrast to $O(n^3)$ for elimination of a full matrix

When the system is banded the work is much the same, but more work is necessary due to the increased number of diagonals.

Diagonal Dominance

This all works nicely if we don't need to pivot. It turns out that this is actually quite common in practice.

Pivoting is not necessary if a matrix A is <u>diagonally dominant</u>

A matrix A is <u>diagonally dominant</u> (or <u>strictly diagonally dominant</u>) if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad (1 \le i \le n)$$

A is <u>weakly diagonally dominant</u> if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad (1 \le i \le n)$$

and

$$|a_{ii}| = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad \text{for some } i$$