HOMEWORK HELP 2 FOR MATH 151

Here we go; the second round of homework help. If there are others you would like to see, let me know!

2.4, 43 and 44

At what points are the functions $f(x)$ and $g(x) = xf(x)$continuous, where

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}?$$

$f(x)$ is not continuous anywhere; for $\epsilon = 1/2$, it is clear that no matter what you choose as a limit (L) for f at any point x that for all δ there are y within δ of x such that $f(y)$ is not within $1/2$ of L, since there are both rational and irrational numbers in any open interval. On the other hand, g is continuous precisely at $x = 0$. It is not continuous anywhere else by using the same argument as above, using $\epsilon = x/2$ at the point x. But at $x = 0$, if we choose $\delta = \epsilon$, then if we place y within δ of 0, certainly $g(y)$ will be within ϵ of 0 as well (since it will be either zero already or smaller than ϵ at any rate).

2.6, 2

Use the intermediate value theorem to show that there is a solution of $x^4 - x - 1 = 0$ in the interval $[-1, 1]$.

Consider that, if $f(x) = x^4 - x - 1$, then $f(-1) = 1$ and $f(1) = -1$. So by the IVT, since f is continuous (a polynomial) we know that if K is between -1 and 1, there is some $c \in (-1, 1)$ such that $f(c) = K$. Choose $K = 0$. Then c for this is precisely the desired root.

2.6, 32

Let n be a positive integer. Prove that if $0 \leq a < b$, then $a^n < b^n$. Further show that every nonnegative real number x has a unique nonnegative nth root $x^{1/n}$.

We assume that $0 \leq a < b$. We wish to show $a^n < b^n$ for all positive n. For $n = 1$ this is given. Further, if we assume it is true for $n = k$, then $a^k < b^k$, so as $a < b$ we have $a^{k+1} < b^{k+1}$ (this does not have to be proved, it would only have to be in 160s). So by induction, since we have shown that the statement for $n = k+1$ follows from $n = k$, we are done. Now consider the function $f(t) = t^n - x$. Then $f(0) \leq 0$ and $f(x+1) > 0$ (this is true by the fact that $1 < 1 + x$ and the thing shown in the first part). Thus, as $f(0) \leq 0 < f(x+1)$ and f is continuous (a polynomial), by the intermediate value theorem there is some $t_0 \in [0, x+1]$ such that $f(t_0) = 0$. Thus $t_0^n = x$, so at least t_0 is a positive nth root. But suppose there were another nth root not equal to t_0; call it s. Then either $s < t$ or $s > t$. But then either $t_0^n > s^n$ or $s^n > t_0^n$; either of these cases implies that in fact s is *not* an nth root, though. So t_0 is unique.

2.6, 35

Prove that the cubic equation $x^3 + ax^2 + bx + c = 0$ has at least one real root.

The key is to see that there is a negative and a positive value of this equation $f(x) = 0$; then as it is a polynomial, hence continuous, by the intermediate value theorem it has a zero. We rewrite f as $x^3(1+\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3})$. Then, by the theorem on products of limits, we see that the limit of the cubic as x approaches $\pm\infty$ is in fact the same as the limit of x^3 (the thing in parentheses certainly goes to 1). Since these limits we know are in fact $\pm\infty$, we know that somewhere there then has to be x, y such that $f(x) > 0$ and $f(y) < 0$; otherwise the limits couldn't tend to infinity. But that is what was desired to use the IVT.

3.1, 11

Differentiate $f(x) = 1/x^2$.

We form $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. The fraction is $\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$, which simplifies to $\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2} = \frac{-2xh-h^2}{hx^2(x+h)^2}$.

Taking the limit and canceling h we get $f'(x) = \frac{-2x}{x^4} = -\frac{2}{x^3}$.

3.1, 14

Find $f'(2)$ using the definition of derivative, for $f(x) = 7x - x^2$.

We form $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. This is, for $x = 2$, $\lim_{h\to 0} \frac{7(2+h)-(2+h)^2-7(2)+2^2}{h}$ which simplifies to $\lim_{h\to 0} 7 - \frac{4h+h^2}{h} = 7 - 4 + 0 = 3$.

3.1, 49

Show that

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x, & x > 1 \end{cases}$$

is not differentiable at $x = 1$.

Note that $f(x) = 1$, but $\lim_{x\to 1^+} = 2$ by continuity of $2x$. Thus f is not continuous, so by Theorem 3.1.4 it certainly is not differentiable.

3.1, 50

Find A and B given that the function

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ Ax + B, & x > 1 \end{cases}$$

is differentiable at $x = 1$.

We need to find out for what values of A, B the function is continuous and has the same derivative from the right and the left (i.e. is differentiable) at $x = 1$. For continuity, clearly the condition $(1)^3 = A + B$ suffices, since the component functions are continuous. For differentiability, the right and left derivatives at 1 would be A and $3(1)^2$ respectively. So we need $A = 3$, which means that $B = -2$.

3.1, 65

Let $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Show that f and g are both continuous at 0, f is *not* differentiable at 0, and g *is* differentiable at 0 (find $g'(0)$).

For the first part, we consider that $-x \leq f(x) \leq x$ and $-x^2 \leq g(x) \leq x^2$ as sin is bounded by ± 1; hence by the pinching theorem, since the limits of $\pm x$ and $\pm x^2$ are zero, so are those of f and g, which proves continuity.

To continue, we examine the defining limits for the two derivatives:

$$f'(x) = \lim_{x\to 0} \frac{h\sin(1/h)}{h} \text{ and } g'(x) = \lim_{x\to 0} \frac{h^2\sin(1/h)}{h}$$

The second one (after cancelling h) is simply the limit of f at 0, so it exists and in fact is zero. The first one, however, is purporting to be $\lim_{x\to 0}\sin(1/x)$, which we have previously shown does not exist. So f is not continuous at zero.

3.2, 3.2.4

We wish to prove that if f_i are differentiable at x for $1 \le i \le n$, and $\alpha_i \in \mathbb{R}$, then $\alpha_1 f_1 + \cdots \alpha_n f_n$ is differentiable at x and

$$(\alpha_1 f_1 + \cdots \alpha_n f_n)'(x) = \alpha_1 f_1'(x) + \cdots \alpha_n f_n'(x).$$

To do this, we proceed by induction. For $n = 1$, this is completely trivial as both sides are the same already. Suppose we have it for $n = k$. Then consider some $(\alpha_1 f_1 + \cdots \alpha_{k+1} f_{k+1})'(x)$; if we construct its derivative to be what we want, then it is differentiable at x since we know its derivative. But let $f(x) = (\alpha_1 f_1 + \cdots \alpha_k f_k)(x)$ and $g(x) = \alpha_{k+1} f_{k+1}(x)$. By Theorem 3.2.3, $(f + g)'(x) = f'(x) + g'(x)$. So here $(\alpha_1 f_1 + \cdots \alpha_{k+1} f_{k+1})'(x) = (\alpha_1 f_1 + \cdots \alpha_k f_k)'(x) + \alpha_{k+1} f_{k+1}'(x)$. But using the induction hypothesis, we see that in fact

$$(\alpha_1 f_1 + \cdots \alpha_{k+1} f_{k+1})'(x) = \alpha_1 f_1'(x) + \cdots \alpha_{k+1} f_{k+1}'(x).$$

So by induction, the statement is true for all n.

3.3, 56

Verify the identity $f(x)g''(x) - f''(x)g(x) = \frac{d}{dx}[f(x)g'(x) - f'(x)g(x)]$.

We simply differentiate on the right. The right side is, by the addition and product rules, $f'(x)g'(x) + f(x)g''(x) - f''(x)g(x) - f'(x)g'(x)$. But this is the left side, after cancellation.

Project 3.3

Show that if $g(x) = [f(x)]^4$, then $g'(x) = 4f^3(x)f'(x)$. Show in general that if $g(x) = [f(x)]^n$ for any integer n, then $g'(x) = kf^{k-1}(x)f'(x)$.

By writing $f^4(x) = f^2(x)f^2(x)$, we see that by the product rule and information in the problem $(f^4(x))' = 2f(x)f'(x)f^2(x) + 2f(x)f'(x)f^2(x)$ which is what we wanted. For positive n, consider that we already know this for $n = 1, 2, 3, 4$ (in fact for $n = 0$, since then it's just 0). Then assume we have it for $n = k$. Then $f^{k+1}(x) = f(x)f^k(x)$, so by the product rule we have $(f^{k+1}(x))' = f'(x)f^k(x) + f(x)(f^k(x))' = (1+k)f'(x)f^k(x)$ as desired. So by induction we have it for positive n. For negative n, we use precisely the same induction argument, except instead of the product rule we use the quotient rule. So we have it for $n = -1$, and then assuming for $n = -k$, we have $(f^{-k-1}(x))' = (\frac{f^{-k}(x)}{f(x)})' =$, which is $\frac{(-k)f'(x)f^{-k-1}(x)f(x) - f^{-k}(x)f'(x)}{f^2(x)}$. We can simplify this to $(-k-1)f'(x)\frac{f^{-k}(x)}{f^2(x)} = (-k-1)f'(x)f^{-k-2}(x)$ as desired. Thus by induction the negative integers also have this property, so they all do.

3.4, 4

Find the rate of change of $y = 1/x$ with respect to x at $x = -1$.

We simply take the derivative! So $y' = 1/x^2$, which at the point indicated is 1.

3.4, 8

Find the rate of change of the surface area with respect to the radius r. What is this rate of change when $r = r_0$? How must r_0 be chosen so that the rate of change is 1?

The surface area is $4\pi r^2$. So we take the derivative with respect to r, which is $8\pi r$. So at $r = r_0$, we have $8\pi r_0$. For this to be 1, we solve $8\pi r_0 = 1$; this yields $r_0 = \frac{1}{8\pi}$.

3.4, 15

For what value of x is the rate of change of $y = ax^2 + bx + c$ with respect to x the same as the rate of change of $z = bx^2 + ax + c$ with respect to x? Assume $a \neq b$.

We take the derivatives $\frac{dy}{dx}$ and $\frac{dz}{dx}$. This is $y' = 2ax + b$, and $z' = 2xb + a$. For these to be equal, $2ax + b = 2xb + a$, which implies $x = \frac{a-b}{2b-2a} = -1/2$. Clearly, if $a = b$ then the derivatives are equal always.

3.4, 39

During what time interval(s) is the particle in motion described by $x(t) = 5t^4 - t^5$ speeding up?

We have seen in the chapter that this is the change of the speed with respect to time, which is the change of the change of position. So we take the *second* derivative. This is $x''(t) = 60t^2 - 20t^3$. We are asked when this is positive. Since it is continuous and the zeros are $t = 0$ and $t = 3$, we use the intermediate value theorem to check the three possible intervals. Since $x''(-1) = 80$, $x''(1) = 40$, and $x''(10) = -14000$, we see it is positive on $(-\infty, 0) \cup (0, 3)$, which is the answer.

3.4, 54

To estimate the height of a bridge a man drops a stone into the water below. How high is the bridge (a) if the stone hits the water 3 seconds later? (b) if the man hears the splash 3 seconds later?

We recall that motion in the vertical direction is described by $y(t) = -16t^2 + v_0 t + y_0$ (for feet and seconds). So, with the height of the bridge being considered y_0 and the initial velocity clearly $v_0 = 0$, then $y(3) = 0$ in part (a). Thus, $-16(3)^2 + y_0 = 0$, or $y_0 = 144$ feet. However, in part (b), the stone falls and then the man hears the splash. Since we are told sound travels at the constant speed of 1080 feet per second, we know that it traveled y_0 feet in a certain amount of time, say s seconds. Then the stone actually fell in only $3 - s$ seconds, and $y(3 - s) = 0$. We wish to find y_0. Then $0 = -16(3 - s)^2 + y_0$. But $1080s = y_0$, so $s = \frac{y_0}{1080}$. Substituting we see that $y_0 = 144 - \frac{96y_0}{1080} + \frac{16y_0^2}{(1080)^2}$. We solve for y_0 and, after using the quadratic formula, arrive at $y_0 = 147/4$ (it turns out the quadratic factors as a square).

3.4, 60

Find the marginal cost function at a production level of 100 units and compare with the actual cost of the 101st unit for $C(x) = 1000 + 2x + 0.02x^2 + 0.0001x^3$.

We have the marginal cost defined as $C'(100)$, which is $2+0.04(100)+0.0003(100)^2 = 2 + 4 + 3 = 9$. On the other hand, $C(101) - C(100) =$

$$1000+2(101)+0.02(101)^2+0.0001(101)^3-(1000+2(100)+0.02(100)^2+0.0001(100)^3)$$

which simplifies to $2 + 0.02(201) + 0.0001(30301) = 9.0501$. So the accuracy of the estimate is to 0.0501, which is around a half of a percent from the actual answer - not bad!

3.5, 12

Differentiate $f(x) = (x + \frac{1}{x})^3$.

By the chain rule, $f'(x) = 3(x + \frac{1}{x})(1 - \frac{1}{x^2})$.

3.5, 28

Find $\frac{dy}{dt}$ if $y = 1 + u^2$, $u = \frac{1-7x}{1+x^2}$, and $x = 5t + 2$.

We use the chain rule; $\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$. This is simply $\frac{dy}{dt} = 2u(\frac{(1+x^2)(-7)-(1-7x)(2x))}{(1+x^2)^2})(5)$. We then write

$$10\frac{1-7(5t+2)}{1+(5t+2)^2}\frac{(1+(5t+2)^2)(-7)-(1-7(5t+2))(2(5t+2)))}{(1+(5t+2)^2)^2}$$

3.6, 2

Differentiate $y = x^2 \sec x$.

Using the quotient rule, since $y = \frac{x^2}{\cos x}$, we have $y'(x) = \frac{2x\cos x + x^2 \sin x}{\cos^2 x}$.

3.6, 58a

Verify that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

We note that $\cot x = \frac{\cos x}{\sin x}$. Then by the quotient rule, $\frac{d}{dx}(\cot x) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x}$, which is just $-\frac{1}{\sin^2 x} = -\csc^2 x$.

3.7, 4

Use implicit differentiation to obtain dy/dx in terms of x and y from the equation $\sqrt{x} + \sqrt{y} = 4$.

Applying d/dx to both sides gives us $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$. Then solving for $\frac{dy}{dx} = -\sqrt{\frac{x}{y}}$.

3.8, 2

A particle is moving in the circular orbit $x^2 + y^2 = 25$. As it passes through $(3, 4)$, its y-coordinate is decreasing at the rate of 2 units per second. At what rate is the x-coordinate changing?

We are asked for dx/dt. By implicit differentiation, we have $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$; further, we are given that $dy/dt = -2$ and $(x, y) = (3, 4)$. Plugging it all in gives $6\frac{dx}{dt} - 16 = 0$, or $dx/dt = 8/3$.

3.8, 7

A heap of rubbish in the shape of a cube is being compacted. Given that the volume decreases at a rate of 2 cubic meters per minute, find the rate of change of an edge of the cube when the volume is exactly 27 meters cubic meters. What is the rate of change of the surface area of the cube at that instant?

We recall the relation between volume, surface area, and length of an edge: $V = l^3$ and $S = 6l^2$ if l is the length. So we are told $dV/dt = -2$, and we want dl/dt at $V = 27$. Clearly at that point $l = 3$, since $3^3 = 27$. Using the chain rule, we see that $\frac{dV}{dt} = 3l^2\frac{dl}{dt}$. Thus at this time, $\frac{dl}{dt} = \frac{dV/dt}{3(3)^2}$, which is $-\frac{2}{27}$. That is, the edge of the cube is slowly getting smaller at this rate. Since the surface area $S = 6l^2$, after another use of the chain rule (or implicit differentiation), $\frac{dS}{dt} = 8l\frac{dl}{dt} = 8(3)(-\frac{2}{27})$. Thus the surface area is changing at the rate of $-\frac{16}{9}$ square feet per minute.

3.8, 17

A 13-foot ladder is leaning against a vertical wall. If the bottom of the ladder is being pulled away from the wall at the rate of 2 feet per second, how fast is the area of the triangle formed by the wall, the ground and the ladder changing at the instant the bottom of the ladder is 12 feet from the wall.

It'll be hard to do this without a diagram, but you all have good imaginations, based on the crazy entrance essays for the U of C.

We have some information right off the bat. If we call the distance of the base from the wall x, then we know $dx/dt = 2$. Very conveniently, when $x = 12$, the height of the wall where the ladder touches it is precisely $\sqrt{169 - 144} = 5$ feet. Now the area $A = \frac{1}{2}hx$, where h is the height. Thus we want $\frac{dA}{dt} = \frac{1}{2}(h\frac{dx}{dt} + x\frac{dh}{dt})$. In general, $\frac{dh}{dt} = -\frac{2x}{\sqrt{169-x^2}}$ as $h = \sqrt{169 - x^2}$. At this point, that means that $\frac{dh}{dt} = -\frac{24}{5}$. So $\frac{dA}{dt} = \frac{1}{2}(5(2) + 12(-\frac{24}{5})) = -23.8$. Looks like the area is decreasing pretty rapidly.

3.8, 34

The diameter and height of a right circular cylinder are found at a certain instant to be 10 centimeters and 20 centimeters, respectively. If the diameter is increasing at the rate of 1 centimeter per second, what change in height will keep the volume constant?

Let's tabulate the information we have thus far. The volume of such a cylinder is $V = \pi(D/2)^2h$; we also know $dD/dt = 1$. We want to keep V constant, so let's regard it as such; then $h = \frac{4V}{D^2\pi}$, so $\frac{dh}{dt} = -\frac{8V}{D^3\pi}$. At the point in question, then, $\frac{dh}{dt} = -\frac{8(\pi(100)/4)}{1000\pi} = -\frac{1}{5}$. So a rate of change of $-\frac{1}{5}$ will suffice.